3.4 "Computing" CT Convolution

For DT systems, conv. is something we do
for analysis and for implementation

For CT systems, we do conv. for analysis
Nature does " " implementation.

CT Conv. Properties

Many of the properties are the same as
for DT Conv.

We only discuss the new ones here.

1. Derivative Property

derivative

$$\frac{d}{dt}[x(t) * v(t)] = \dot{x}(t) * v(t)$$

$$= x(t) * \dot{v}(t)$$

2. Integration Property

Let $y(t) = x(t) * h(t)$

$$\int_{-\infty}^{t} y(\lambda)d\lambda = \left[\int_{-\infty}^{t} x(\lambda)d\lambda\right] * h(t) = x(t) * \left[\int_{-\infty}^{t} h(\lambda)d\lambda\right]$$

Steps for Graphical Convolution: $y(t) = x(t)*h(t)$

Note: I use τ for what the book uses λ... It is not a big deal as they are just dummy variables!!!

1. **Re-Write the signals as functions of τ**: $x(\tau)$ and $h(\tau)$

2. **Flip** just one of the signals around $t = 0$ to get either $x(-\tau)$ or $h(-\tau)$
   a. It is usually best to flip the signal with shorter duration
   b. For notational purposes here: we'll flip $h(\tau)$ to get $h(-\tau)$

3. **Find Edges** of the flipped signal
   a. Find the left-hand-edge τ-value of $h(-\tau)$: **call it $\tau_{L,0}$**
   b. Find the right-hand-edge τ-value of $h(-\tau)$: **call it $\tau_{R,0}$**

4. **Shift** $h(-\tau)$ by an arbitrary value of t to get $h(t - \tau)$ and **get its edges**
   a. Find the left-hand-edge τ-value of $h(t - \tau)$ as a function of t: **call it $\tau_{L,t}$**
      - **Important:** It will *always* be… $\tau_{L,t} = t + \tau_{L,0}$

   b. Find the right-hand-edge τ-value of $h(t - \tau)$ as a function of t: **call it $\tau_{R,t}$**
      - **Important:** It will *always* be… $\tau_{R,t} = t + \tau_{R,0}$

Note: If the signal you flipped is NOT finite duration,
one or both of $\tau_{L,t}$ and $\tau_{R,t}$ will be infinite ($\tau_{L,t} = -\infty$ and/or $\tau_{R,t} = \infty$)

Steps Continued

5. <u>Find Regions of τ-Overlap</u>

a. What you are trying to do here is find intervals of t over which the product $x(\tau)\,h(t - \tau)$ has a single mathematical form in terms of τ

b. In each region find: Interval of t that makes the identified overlap happen

c. Working examples is the best way to learn how this is done

<u>Tips</u>: Regions should be contiguous with no gaps!!!

Don't worry about $<$ vs. $\leq$ etc.

6. For Each Region: **<u>Form the Product $x(\tau)\,h(t - \tau)$ and Integrate</u>**

a. <u>Form product</u> $x(\tau)\,h(t - \tau)$

b. <u>Find the Limits of Integration</u> by finding the interval of τ over which the product is nonzero

   i. Found by seeing where the edges of $x(\tau)$ and $h(t - \tau)$ lie

   ii. Recall that the edges of $h(t - \tau)$ are $\tau_{L,t}$ and $\tau_{R,t}$, which often depend on the value of t

      - So… the limits of integration <u>may</u> depend on t

c. <u>Integrate the product</u> $x(\tau)\,h(t - \tau)$ over the limits found in 6b

   i. The result is generally a function of t, but is only valid for the interval of t found for the current region

   ii. Think of the result as a "time-section" of the output $y(t)$

Steps Continued

7. **<u>“Assemble” the output</u>** from the output time-sections for all the regions

   a. Note: you do NOT add the sections together

   b. You define the output “piecewise”

   c. Finally, if possible, look for a way to write the output in a simpler form

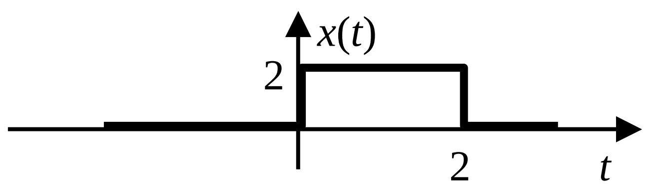
Example: Graphically Convolve Two Signals

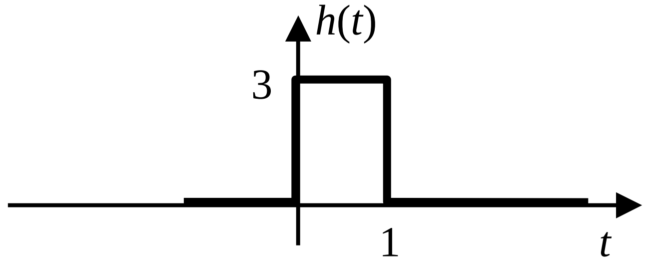
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

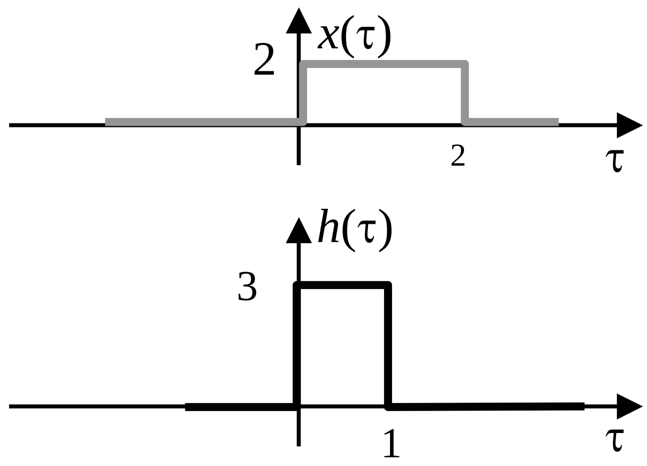
We'll see later that these two forms are Equal

<u>This</u> is why we can flip <u>either</u> signal

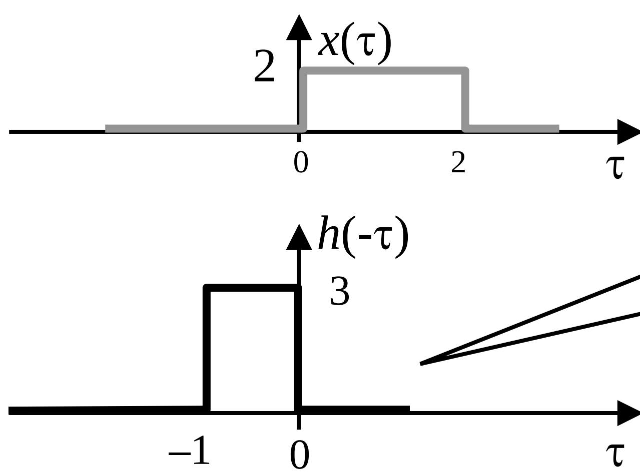




Step #1: Write as Function of τ

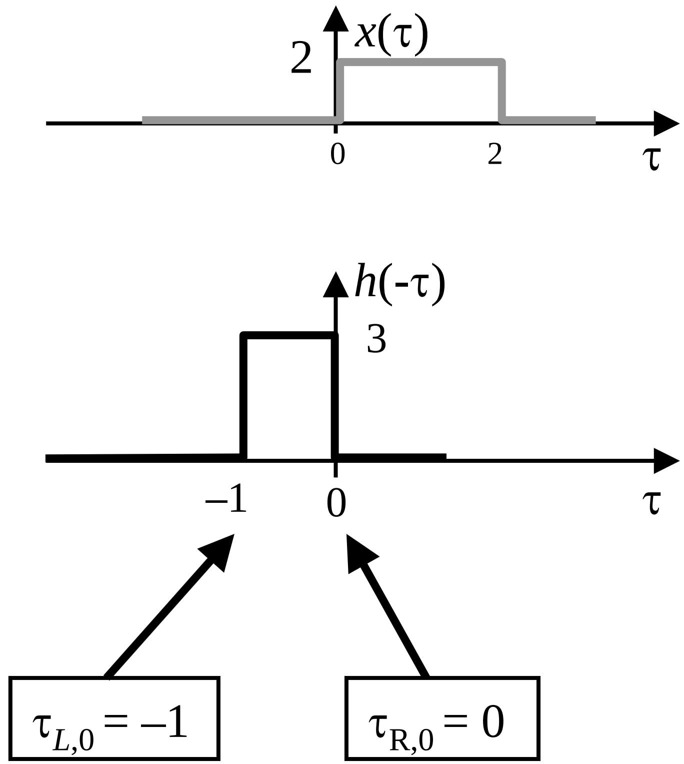


Step #2: Flip *h*(τ) to get *h*(-τ)



Usually Easier to Flip the Shorter Signal

Step #3: Find Edges of Flipped Signal



Motivating Step #4: Shift by *t* to get $h(t\text{-}\tau)$ & Its Edges

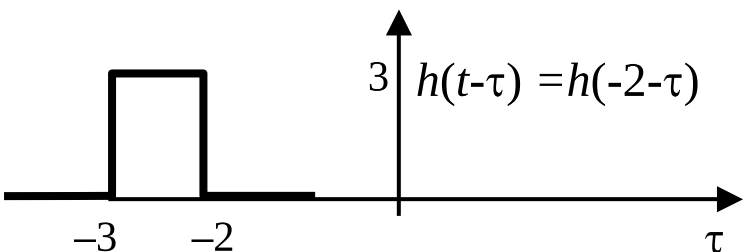
Just looking at 2 "arbitrary" *t* values

In Each Case We Get

$$\tau_{L,t} = t + \tau_{L,0}$$

$$\tau_{R,t} = t + \tau_{R,0}$$

For t = -2



$\tau_{L,t} = \boldsymbol{t} + \tau_{L,0}$

$\tau_{L,t} = \boldsymbol{t - 1}$

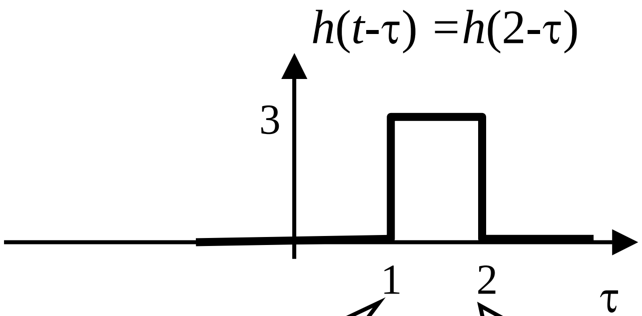
$\tau_{L,-2} = \mathbf{-2 - 1}$

$\tau_{R,t} = \boldsymbol{t} + \tau_{R,0}$

$\tau_{R,t} = \boldsymbol{t} + \mathbf{0}$

$\tau_{R,-2} = -2 + \mathbf{0}$

For t = 2



$\tau_{L,t} = \boldsymbol{t} + \tau_{L,0}$

$\tau_{L,t} = \boldsymbol{t - 1}$

$\tau_{L,2} = \mathbf{2 - 1}$

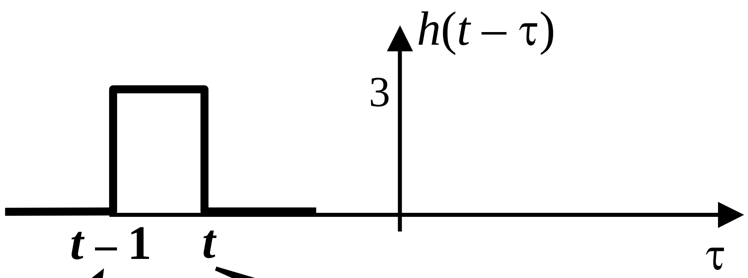
$\tau_{R,t} = \boldsymbol{t} + \tau_{R,0}$

$\tau_{R,t} = \boldsymbol{t} + \mathbf{0}$

$\tau_{R,2} = 2 + \mathbf{0}$

Doing Step #4: Shift by *t* to get *h*(*t*-τ) & Its Edges

For *Arbitrary* Shift by *t*



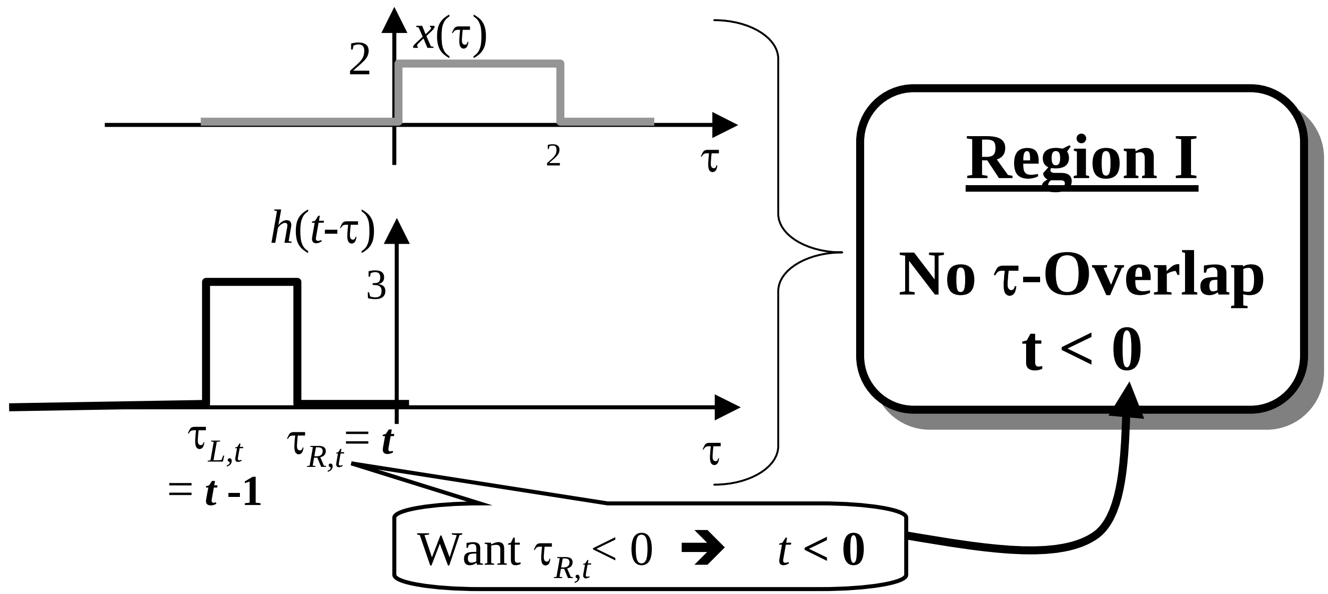
$\tau_{L,t} = \boldsymbol{t} + \tau_{L,0}$

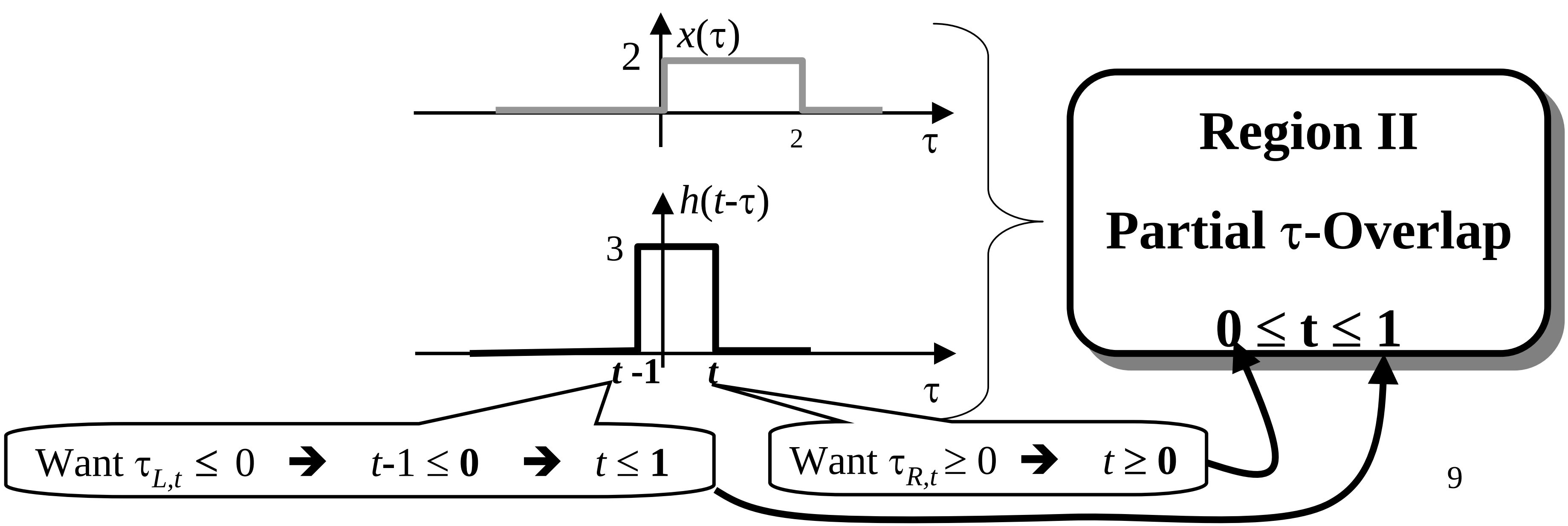
$\tau_{L,t} = \boldsymbol{t - 1}$

$\tau_{R,t} = \boldsymbol{t} + \tau_{R,0}$

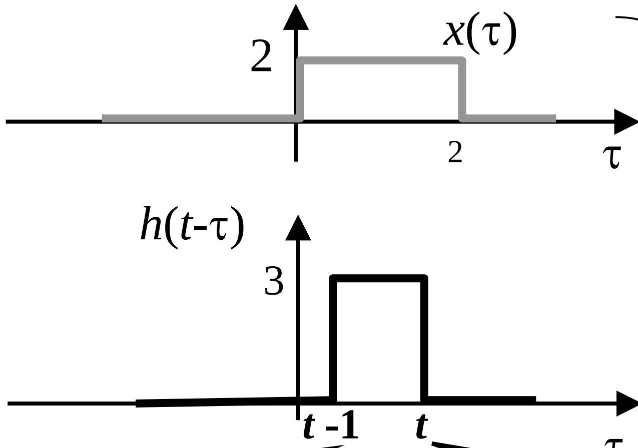
$\tau_{R,t} = \boldsymbol{t + 0}$

Step #5: Find Regions of τ-Overlap





Step #5 (Continued): Find Regions of τ-Overlap



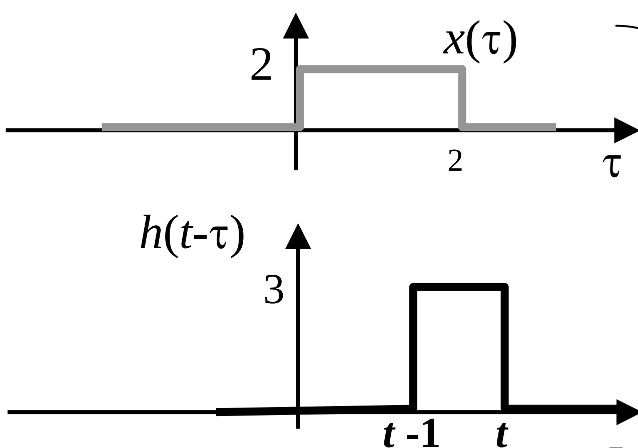
Region III

Total τ-Overlap

$\mathbf{1 < t \leq 2}$

Want $\tau_{L,t} > 0$ ➔ t-1 > **0** ➔ **t > 1**

Want $\tau_{R,t} \leq 2$ ➔ **t ≤ 2**



Region IV

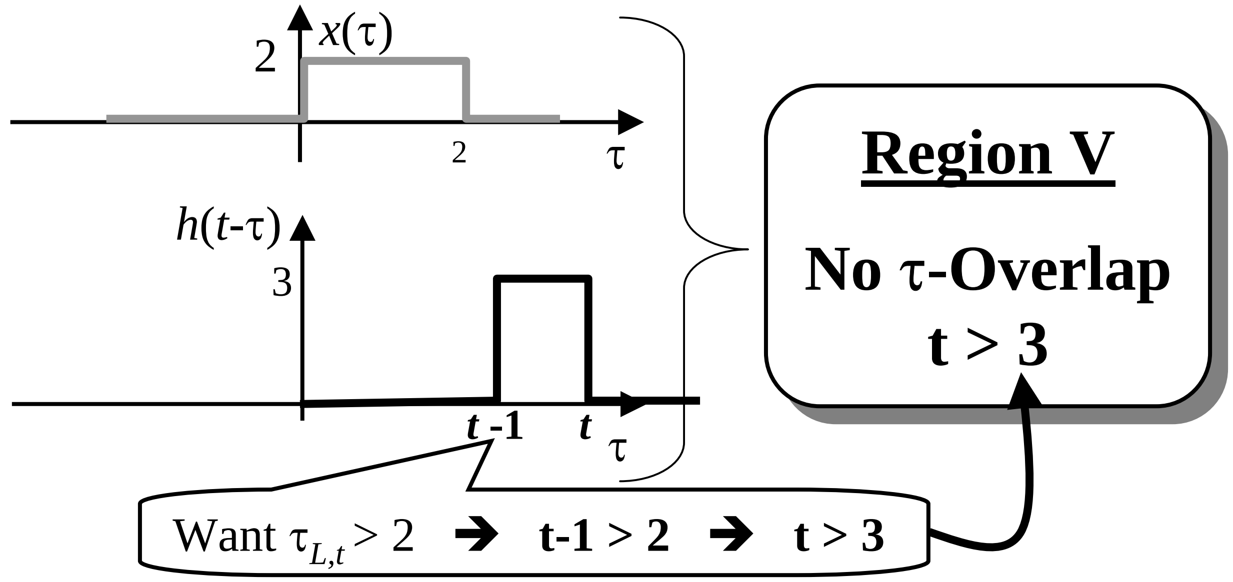
Partial τ-Overlap

$\mathbf{2 < t \leq 3}$

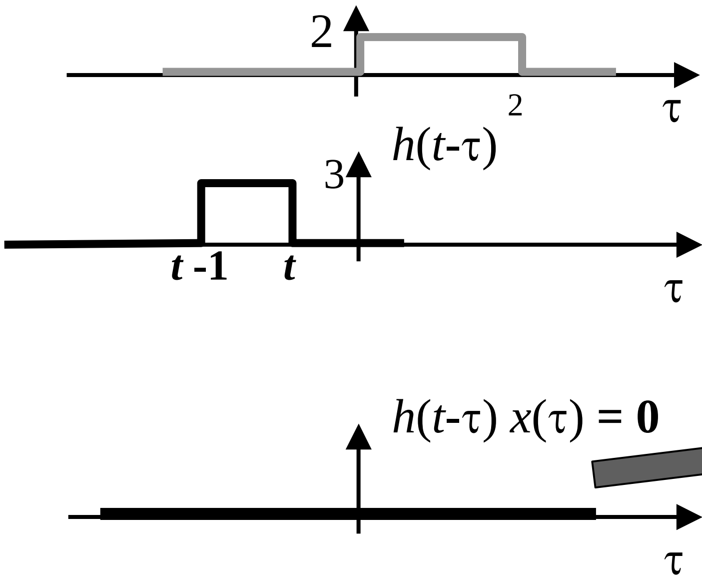
Want $\tau_{L,t} \leq 2$ ➔ t-1 ≤ 2 ➔ **t ≤ 3**

Want $\tau_{R,t} > 2$ ➔ **t > 2**

Step #5 (Continued): Find Regions of τ-Overlap



Step #6: Form Product & Integrate For Each Region



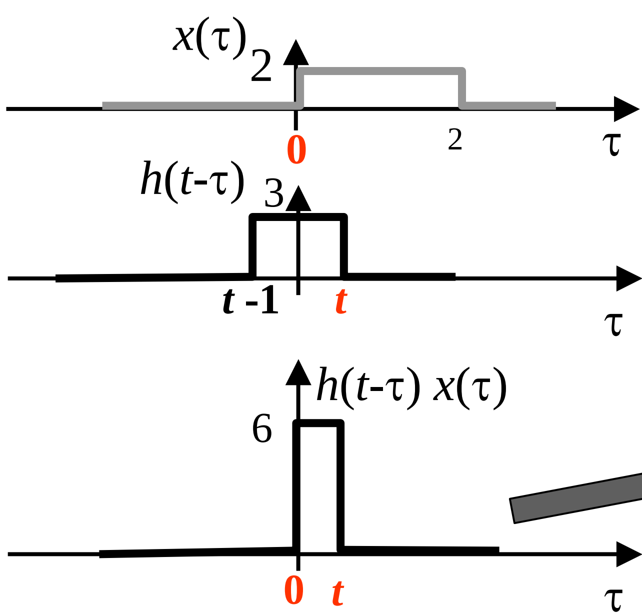
Region I: t < 0

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} 0 d\tau = 0$$

$$y(t) = 0 \quad \text{for all} \quad t < 0$$

With 0 integrand the limits don't matter!!!



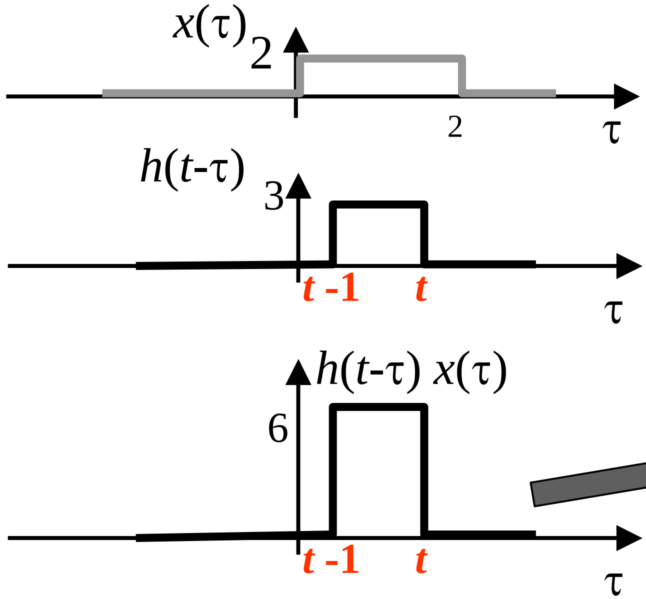
Region II: 0 ≤ t ≤ 1

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{0}^{t} 6 d\tau = [6\tau]_0^t = 6t - 6 \times 0 = 6t$$

$$y(t) = 6t \quad \text{for} \ \ 0 \le t \le 1$$

Step #6 (Continued): Form Product & Integrate For Each Region

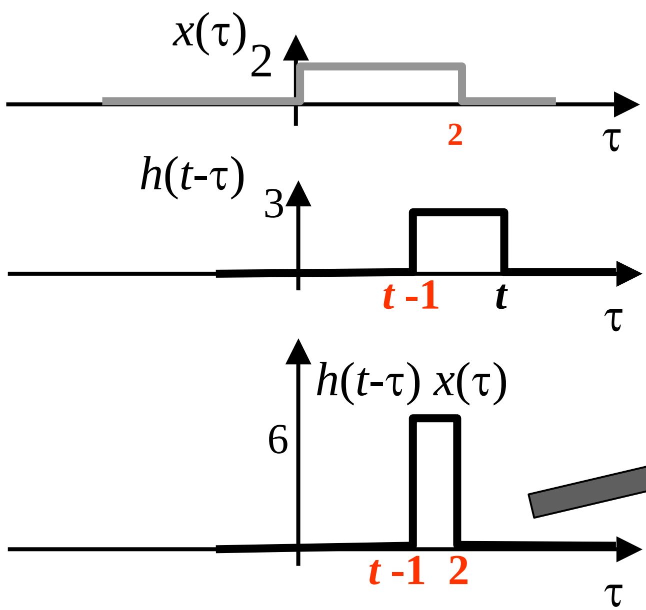


Region III: $1 < t \le 2$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{t-1}^{t} 6d\tau = [6\tau]_{t-1}^{t} = 6t - 6(t-1) = 6$$

$y(t) = 6$ for all t such that: $1 < t \le 2$



Region IV: $2 < t \le 3$

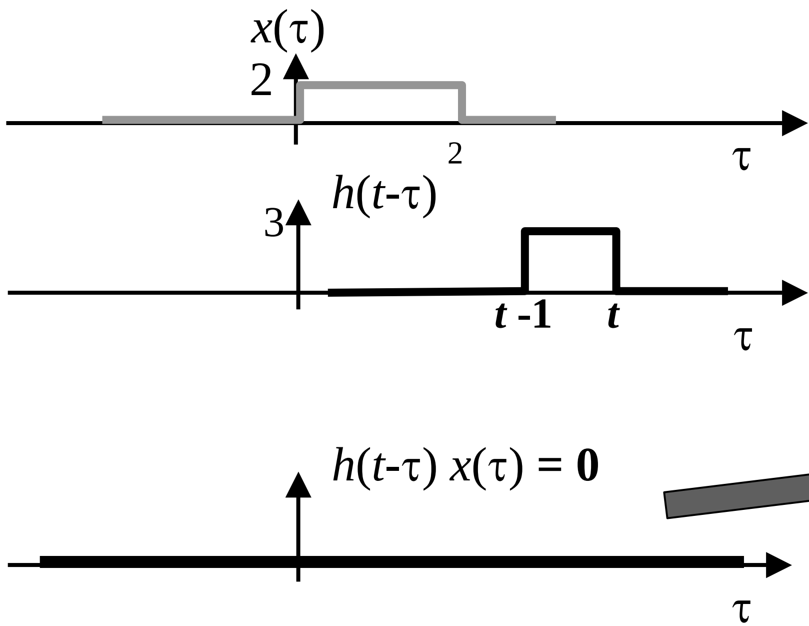
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{t-1}^{2} 6d\tau = [6\tau]_{t-1}^{2} = 6\times 2 - 6(t-1) = -6t + 18$$

$y(t) = -6t + 18$ for $2 < t \le 3$

Step #6 (Continued): Form Product & Integrate For Each Region

Region V: t > 3



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} 0 d\tau = 0$$

$$y(t) = 0 \quad \text{for all} \quad t > 3$$

Step #7: "Assemble" Output Signal

Region I	Region II	Region III	Region IV	Region V
$t < 0$	$0 \le t \le 1$	$1 < t \le 2$	$2 < t \le 3$	$t > 3$
$y(t) = 0$	$y(t) = 6t$	$y(t) = 6$	$y(t) = -6t + 18$	$y(t) = 0$