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Optimal communication scheduling and remote estimation over an additive noise channel☆

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ABSTRACT

This paper considers a sequential sensor scheduling and remote estimation problem with one sensor and one estimator. The sensor makes sequential observations about the state of an underlying memoryless stochastic process, and makes a decision as to whether or not to send this measurement to the estimator. The sensor and the estimator have the common objective of minimizing expected distortion in the estimation of the state of the process, over a finite time horizon. The sensor is either charged a cost for each transmission or constrained on transmission times. As opposed to the prior work where communication between the sensor and the estimator was assumed to be perfect (noiseless), in this work an additive noise channel with fixed power constraint is considered; hence, the sensor has to encode its message before transmission. Under some technical assumptions, we obtain the optimal encoding and estimation policies within the piecewise affine class in conjunction with the optimal transmission schedule. The impact of the presence of a noisy channel is analyzed numerically based on dynamic programming. This analysis yields some rather surprising results such as a phase-transition phenomenon in the number of used transmission opportunities, which was not encountered in the noiseless communication setting.

1. Introduction

The communication scheduling and remote state estimation problem arises in the applications of wireless sensor networks, such as environmental monitoring and networked control systems. As an example of environmental monitoring, researchers at the National Aeronautics and Space Administration (NASA) Earth Science group are interested in monitoring the evolution of the soil moisture, which is used in weather forecast, ecosystem process simulation, etc. (Shuman et al., 2010). In order to achieve that goal, the sensor networks are built over an area of interest. The sensors collect data on the soil moisture and send them to the decision unit at NASA via wireless communication. The decision unit at NASA forms estimates on the evolution of the soil moisture based on the messages received from the sensors. Similarly, in networked control systems, where the objective is to control some remote plants, sensor networks are built to measure the states of the remote plants. Sensors transmit their measurements to the controller via a wireless communication network, and the controller estimates the state of the remote plant and generates a control signal based on that estimate (Hespanha, Naghshtabrizi, & Xu, 2007). In both scenarios, the quality of the remote state estimation strongly affects the quality of decision making at the remote site, that is, weather prediction or control signal generation. The networked sensors are usually constrained by limits on power (Akyildiz, Su, Sankarasubramaniam, & Cayirci, 2002). They are not able to communicate with the estimator at every time step and thus, the estimator has to produce its best estimate based on the partial information received from the sensors. Therefore, the communication between the sensors and the estimator should be scheduled judiciously, and the estimator should be designed properly, so that the state estimation error is minimized under the communication constraints.

Research on the general sensor scheduling problem dates back to the 1970s. In one of the earliest works (Athans, 1972), the problem formulation is such that only one out of several sensors can be selected at each instant of time to observe the output of a linear stochastic system. Using the measurements over a finite time interval, the goal is to form prediction on some future state of the

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system. Furthermore, each sensor is associated with a certain measurement cost. The author proposed an off-line deterministic sensor scheduling strategy that minimizes the sum of measurement cost over the time interval and prediction error. Gupta, Chung, B. Hassibi, and Murray (2006) studied the sensor scheduling problem over infinite time horizon. Similar to the problem in Athans (1972), only one sensor can be selected at each instant of time. However, there is no measurement cost associated with each sensor. The authors proposed an off-line stochastic sensor scheduling strategy such that the expected steady state estimation error is minimized. Yang and Shi (2011) studied the off-line sensor scheduling problem where there is only one sensor observing the state of a linear stochastic system. The sensor can communicate with the remote estimator only a limited number of times. The objective was to minimize the cumulative estimation error over a finite time horizon. It was shown that the optimal sensor scheduling strategy is to distribute the limited communication opportunities uniformly over the time horizon. The authors of the papers discussed above considered off-line sensor scheduling problems. "Off-line sensor scheduling" means the sensor is scheduled to take observation or conduct communication based on some a priori information about the system (e.g. statistics of random variables, system matrices). The on-line information (e.g. sensor's observation, battery's energy level) is not taken into account when making schedules. Some other selected work on off-line sensor scheduling problems can be found in Mo, Garone, Casavola, and Sinopoli (2011), Ren, Cheng, Chen, Shi, and Sun (2013) and Shi and Zhang (2012).

With the advances in hardware devices, sensors are endowed with stronger computational capabilities. Consequently, the sensors are able to make schedules based on all the information they have (a priori information as well as on-line information), which motivates the formulation of on-line sensor scheduling problems. Åström and Bernhardsson (2002) considered a state estimation problem with a first-order stochastic system. They compared the estimation error over infinite time horizon obtained by periodic sampling and threshold event-triggered sampling. The periodic sampling is one of the off-line sensor scheduling strategies while the threshold event-triggered sampling is one of the on-line sensor scheduling strategies. They showed that the threshold event-triggered sampling, which is also called "threshold-based communication strategy", leads to better performance in state estimation compared with periodic sampling. The global optimality of threshold-based communication strategy in this context is proved later by Nar and Başar (in press). Imer and Başar (2010) considered the on-line sensor scheduling and remote state estimation problem over a finite time horizon. In the formulation, the sensor is restricted to communicate only a limited number of times. By considering the communication strategies within the class of threshold-based strategies, the paper has shown that there exists a unique threshold-based communication strategy achieving the best performance on remote state estimation. Furthermore, the optimal threshold can be computed by solving a dynamic programming equation. Bommannavar and Başar (2008) later extended the result of Imer and Başar (2010) to multi-dimensional systems. The continuous-time version of the problem in Imer and Başar (2010) has been studied by Rabi, Moustakides, and Baras (2006). Xu and Hespanha (2004) considered the networked control problem involving state estimation and communication scheduling, which can be viewed as a sensor scheduling and remote estimation problem. They fixed the estimator to be Kalman-like and designed an event-triggered sensor that minimizes the time average of the sum of the communication cost and estimation error over infinite time horizon. They showed that the optimal communication strategy is deterministic and stationary, and is a function of the estimation error. Wu, Jia, Johansson, and Shi (2013) considered the sensor scheduling and estimation problem subject to constraints on the average communication rate over infinite time horizon. The authors assumed that the sensor has noisy observations on the system state. By restricting the sensor scheduling strategies to the threshold event-triggered class, they derived the exact minimum mean square error (MMSE) estimator. However, the exact MMSE estimator is nonlinear and thus computationally intractable. Under a Gaussian assumption on the a priori distribution, the authors derived an approximate MMSE estimator, which is Kalman-like. Based on the approximated MMSE estimator, the authors derived conditions on the thresholds so that the average sensor communication rate will not exceed its upper bound. You and Xie (2013) extended the work in Wu et al. (2013) by deriving conditions on the thresholds so that the estimator is stable. Han, Mo, Wu, Weerakkody, Sinopoli, and Shi (2015) showed that if the sensor is fixed to apply some stochastic event-triggered strategy, then the exact MMSE estimator is Kalman-like. Other selected work on remote estimation with event-based sensor operations can be found in Shi, Elliott, and Chen (2016) and Weerakkody, Mo, Sinopoli, Han, and Shi (2013). The work in Han et al. (2015), Wu et al. (2013) and You and Xie (2013) can also be viewed as Kalman-filtering with scheduled observations, which is related to Kalman-filtering with intermittent observations studied in Sinopoli, Schenato, Franceschetti, Poolla, Jordan, and Sastry (2004) and You, Fu, and Xie (2011).

The approaches of Wu et al. (2013) and Xu and Hespanha (2004) involved fixing the communication strategies or estimation strategies to be of a certain type and then deriving the corresponding optimal estimation strategies and communication strategies, respectively. The approach of Imer and Başar (2010), on the other hand, is to derive the jointly optimal communication strategies and estimation strategies. Similarly, Lipsa and Martins (2011) considered the sensor scheduling and remote estimation problem where the sensor is not constrained by communication times but is charged a communication cost. They proposed a threshold event-triggered sensor and a Kalman-like estimator and proved that the proposed sensor and estimator are jointly optimal, minimizing the sum of communication cost and estimation error over a finite time horizon. Nayyar, Başar, Teneketzis, and Veeravalli (2013) considered a similar problem where the sensor is equipped with an energy harvesting sensor. In the work of Nayyar et al. (2013), the problem formulation is such that the sensor is constrained by the energy level of the battery and is also charged a communication cost. It is shown in Nayyar et al. (2013) that an energy dependent threshold event-triggered sensor and a Kalman-like estimator are jointly optimal. Hence, the result of Nayyar et al. (2013) can be viewed as generalization of the results of Imer and Başar (2010) and Lipsa and Martins (2011). In both Lipsa and Martins (2011) and Nayyar et al. (2013), majorization theory was used to prove the optimality of the respective results, which is closely related to the approach in Hajek, Mitzel, and Yang (2008).

It is worth drawing attention to the two different types of constraints that arise in the works mentioned above – hard and soft constraints – as featured in the problem setups of Imer and Başar (2010) and Lipsa and Martins (2011). In the problem of Imer and Başar (2010), the sensor can only communicate for a prespecified number of times. Such a communication constraint is called *hard constraint*. In the work of Lipsa and Martins (2011), however, the sensor is charged a communication cost. This kind of communication constraint is called *soft constraint*. In the problem with hard constraint, the communication strategy must take the remaining communication opportunities as a variable and schedule no communication if there is no remaining opportunity. Such communication strategies guarantee that the number of transmissions made over the time horizon of interest will not exceed the given constraint. In the problem with soft constraint, however, the sensor is not constrained by the number of transmissions, therefore the communication strategy need not take the remaining communication opportunities (which is always equal to the remaining time steps) as a variable. Therefore, the results obtained in one problem cannot be applied directly to the other. For example, if we apply the communication strategy obtained in Lipsa and Martins (2011) to the problem of Imer and Başar (2010), then there exists a positive probability that the sensor decides to communicate at every instant of time, which certainly would violate the hard constraint on communication times. A detailed discussion on the difference between optimization problems with soft constraints and hard constants can be found in Gupta, Langbort, and Başar (2010) and Gupta, Nayyar, Langbort, and Başar (2012).

In this paper, we extend the lines of research in Imer and Başar (2010), Lipsa and Martins (2011) and Nayyar et al. (2013). In previous works, the communication between the sensor and the estimator was assumed to be perfect (no channel noise), which may not be that realistic, even though it was an important first step. This paper investigates the effect of communication channel noise on the design of optimal sensor scheduling and remote estimation strategies. The paper consists of two parts: In the first part, we consider a discrete time sensor scheduling and remote estimation problem with a soft constraint. At each time step, the sensor makes a perfect observation on the state of an independent identically distributed (i.i.d.) source. Next, the sensor decides whether to transmit its observation to the remote estimator or not. The sensor has a soft communication constraint (i.e., the sensor is charged a cost for each transmission). Since the communication channel is noisy, the sensor encodes the message before transmitting it to the estimator. The remote estimator generates a real-time estimate on the state of the source based on the noise corrupted messages received from the sensor. The estimator is charged for estimation error. Our goal is to design the communication scheduling strategy and encoding strategy for the sensor, and the estimation strategy (decoding strategy) for the estimator, to minimize the expected value of the sum of communication cost and estimation cost over the time horizon. Our solution consists of a threshold-based communication scheduling strategy, and a pair of piecewise linear encoding/decoding strategies. We prove optimality under some technical assumptions. Then, we extend the result to the problem with a hard constraint. Using a dynamic programming approach, we obtain the optimal communication scheduling, encoding and decoding strategies. Beyond the qualitatively expected results, we notice some rather surprising effects of the noisy communication considerations in this class of remote estimation problems. For example, over a time horizon T and with a hard transmission limit, $N \leq T$, if the state realizations were so that at time step K, the sensor has used only $N - T + K$ transmissions out of N, the intuitively appealing solution to the noiseless variation of the problem was to transmit from that time on all the observed state realizations without any thresholding, i.e., the threshold is effectively set to zero for samples at time steps $K + 1, \ldots, T$. However, in the noisy setting, we have noticed that this is not the case, the sensor may not use all the transmission opportunities left. This is due to the fact that threshold information – that is whether or not the state sample belongs to an interval – may be more valuable than a "noisy" observation of the state. In fact, depending on the signal-to-noise ratio (SNR) of the channel, there is a fixed number of useful (in average) number of transmissions, and allowing transmissions more than this number, on the average, does not help decrease the expected mean square error (MSE). In all, the major contributions of this paper are as follows:

(1) We formulate two optimization problems involving an additive noise channel under two types of communication constraints.
(2) We show that if the source and noise processes are i.i.d., then the optimization problem with soft constraint can be simplified to a single-stage problem. Furthermore, the optimization problem with hard constraint can be converted to a single-stage problem with soft constraint.
(3) Under some technical assumptions, we show that the optimal communication scheduling policy is a threshold-based one with a unique optimal threshold.
(4) We generate numerical results for the problem with hard constraint. We uncover two surprising facts: first, the optimal estimation error over the time horizon remains constant if the number of communication opportunities exceeds some threshold. In other words, the communication opportunities above the threshold are redundant in terms of reducing the estimation error. Second, the sensor may not use all the communication opportunities by the end of the time horizon. We also analyze the reasons for the occurrence of these two interesting phenomena.

The remainder of the paper is organized as follows: in Section 2, we formulate the optimization problems with soft/hard constraints. In Sections 3 and 4, we present the main results for problems with soft/hard constraints. In Section 5, we present some numerical results for the problem with hard constraint. Finally, in Section 6, we draw concluding remarks and discuss future work.

The authors have five conference papers on the general topic of this paper, listed as Gao, Akyol, and Başar (2015a, b, 2016a, b, c). The specific topics and results of the last three, that is Gao et al. (2016a)–Gao et al. (2016c), are beyond the scope of this paper, as explained in Section 6 (Conclusions). The first two, that is (Gao et al., 2015a, b), have some overlap as far as the problem formulations go, but the current paper substantially improves upon the results in these two papers, as explained in Remark 8 in Section 4.

2. Problem formulation

2.1. System model

Consider, as depicted in Fig. 1, a discrete time communication scheduling and remote estimation problem over a finite-time horizon, that is, $t = 1, 2, \ldots, T$. There is one sensor, one encoder and one remote estimator (which is also called "decoder"). A source process $\{X_t\}$ is a one-dimensional, independent, and identically distributed (i.i.d.) stochastic process, which has density p_X. At time t, the sensor observes X_t. Since the sensor is assumed to have communication constraint (which will be introduced later), it needs to decide whether or not to transmit its observation. Let $U_t \in \{0, 1\}$ be the sensor's decision at time t, where $U_t = 1$ stands for transmission and $U_t = 0$ stands for no transmission. The communication channel is assumed to be noisy. Hence, if the sensor decides to transmit its observation, it sends X_t to the encoder. If the sensor decides not to transmit, it does not send anything to the encoder but a free symbol ϵ stands for its decision. Denote by $\tilde{X}_t$ the message received by the encoder; then

$$\tilde{X}_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \epsilon, & \text{if } U_t = 0. \end{cases}$$

If the encoder receives X_t from the sensor, it sends an encoded message Y_t to the communication channel. The encoder operates under the average power constraint: $\mathbb{E}[Y_t^2 | U_t = 1] \leq P_T$, where the expectation is taken over Y_t. Furthermore, P_T is known and is invariant of time. The encoded message Y_t is corrupted by an additive channel noise V_t. The noise process $\{V_t\}$ is a one-dimensional i.i.d. stochastic process with density p_V, which is independent of $\{X_t\}$. When sending Y_t to the communication channel, the encoder is able to transmit the sign of X_t to the decoder via a side channel,

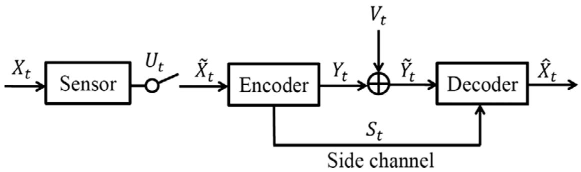

Fig. 1. System model.

which is assumed to be noise-free. If the encoder receives ϵ from the sensor, it sends zero to both the communication channel and the side channel. Consequently, the decoder can deduce the sensor's decision from the message conveyed via the side channel . We use $\tilde{Y}_t$ and S_t to denote the messages received by the decoder from the communication channel and the side channel, respectively, that is

$$\tilde{Y}_t = \begin{cases} Y_t + V_t, & \text{if } U_t = 1 \\ V_t, & \text{if } U_t = 0 \end{cases} \qquad S_t = \begin{cases} \text{sgn}(X_t), & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0. \end{cases}$$

After receiving $\tilde{Y}_t$ and S_t, the decoder produces an estimate on X_t, denoted by $\hat{X}_t$. The decoder is charged for distortion in estimation. We assume that the distortion function $\rho(X_t, \hat{X}_t)$ is the squared error, $(X_t - \hat{X}_t)^2$.

2.2. Communication constraint

The sensor is said to have a *soft constraint* if there is a nonnegative cost function associated with U_t, denoted by $\mathcal{C}(U_t)$. Here, the cost function is assumed to have the form of

$$\mathcal{C}(U_t) = cU_t = \begin{cases} 0, & \text{if } U_t = 0 \\ c, & \text{if } U_t = 1, \end{cases}$$

where c is called the communication cost ($c > 0$), which is known and is invariant of time. The sensor is said to have a *hard constraint* if it is restricted to use the noisy channel for no more than N times ($N < T$).

2.3. Decision strategies

Assume that at time t, the sensor has memory on all its observations up to t, denoted by $X_{1:t}$, and all the decisions it has made up to $t-1$, denoted by $U_{1:t-1}$. The sensor determines whether or not to transmit its observation at time t, based on its current information $(X_{1:t}, U_{1:t-1})$, namely

$$U_t = f_t(X_{1:t}, U_{1:t-1}),$$

where f_t is the communication scheduling policy at time t, and $\mathbf{f} = \{f_1, f_2, \dots, f_T\}$ is the communication scheduling strategy.

Similarly, at time t, the encoder has memory on all the messages received from the sensor up to t, denoted by $\tilde{X}_{1:t}$, and all the messages it has sent to the communication channel and the side channel up to $t-1$, denoted by $Y_{1:t-1}$ and $S_{1:t-1}$, respectively. The encoder generates the encoded message at time t, based on its current information $(\tilde{X}_{1:t}, Y_{1:t-1}, S_{1:t-1})$, namely

$$Y_t = g_t(\tilde{X}_{1:t}, Y_{1:t-1}, S_{1:t-1}),$$

where g_t is the encoding policy at time t, and $\mathbf{g} = \{g_1, g_2, \dots, g_T\}$ is the encoding strategy.

Finally, we assume that at time t, the decoder has memory on all the messages received from the communication channel up to t, denoted by $\tilde{Y}_{1:t}$, and all the messages received from the side channel up to t, which are $S_{1:t}$. The decoder generates the estimate at time t, based on its current information $(\tilde{Y}_{1:t}, S_{1:t})$, namely

$$\hat{X}_t = h_t(\tilde{Y}_{1:t}, S_{1:t}),$$

where h_t is the decoding policy at time t, and $\mathbf{h} = \{h_1, h_2, \dots, h_T\}$ is the decoding strategy.

Remark 1. Although we do not assume that the decoder has memory on its previous estimates up to t, yet it can deduce them from $(\tilde{Y}_{1:t-1}, S_{1:t-1})$ and $h_1, h_2, \dots, h_{t-1}$.

For simplicity, we call the sensor, the encoder, and the decoder as *decision makers*. Correspondingly, we call the communication scheduling policy (strategy), encoding policy (strategy), and decoding policy (strategy) as *decision policies (strategies)*.

2.4. Optimization problem

Consider the settings described above, with the time horizon T, the probability density functions p_X and p_V, and the power constraint P_T as given.

Optimization problem with soft constraint: Given the communication cost c, determine $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ minimizing the functional

$$J(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \mathbb{E}\left\{\sum_{t=1}^{T} cU_t + (X_t - \hat{X}_t)^2\right\}.$$

Optimization problem with hard constraint: Given the number of transmission opportunities N, determine $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ minimizing, under the hard constraint, the cost functional

$$J(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \mathbb{E}\left\{\sum_{t=1}^{T} (X_t - \hat{X}_t)^2\right\}.$$

3. Optimization problem with soft constraint

To begin with, we show that the optimization problem with soft constraint can be simplified to a *single-stage* problem, as described in Theorem 1.

Theorem 1. *Consider the optimization problem in* Section 2.4 *with the soft constraint.*

(1) *Without loss of optimality, one can restrict all the decision makers to apply the decision policies (f_t, g_t, h_t) in the forms of*

$$U_t = f_t(X_t),\ Y_t = g_t(\tilde{X}_t),\ \hat{X}_t = h_t(\tilde{Y}_t, S_t). \tag{1}$$

(2) *Without loss of optimality, one can restrict all the decision makers to apply stationary decision strategies $(\mathbf{f}, \mathbf{g}, \mathbf{h})$, i.e., $\mathbf{f} = \{f, f, \dots, f\}$, $\mathbf{g} = \{g, g, \dots, g\}$, and $\mathbf{h} = \{h, h, \dots, h\}$.*

Before proving Theorem 1, we first introduce the following notations. For any $1 \le a \le b \le T$, let $\mathbf{f}_{a:b}, \mathbf{g}_{a:b}, \mathbf{h}_{a:b}$ denote the subsets of $\mathbf{f}, \mathbf{g}, \mathbf{h}$ such that $\mathbf{f}_{a:b} = \{f_a, f_{a+1}, \dots, f_b\}$, $\mathbf{g}_{a:b} = \{g_a, g_{a+1}, \dots, g_b\}$, and $\mathbf{h}_{a:b} = \{h_a, h_{a+1}, \dots, h_b\}$. Furthermore, let I_{st}, I_{et}, I_{dt} denote the information about the past system states available to the sensor, the encoder, and the decoder, respectively, at time t ($t > 1$), i.e., $I_{st} = \{X_{1:t-1}, U_{1:t-1}\}$, $I_{et} = \{\tilde{X}_{1:t-1}, Y_{1:t-1}, S_{1:t-1}\}$, and $I_{dt} = \{\tilde{Y}_{1:t-1}, S_{1:t-1}\}$. Let I_t be union of I_{st}, I_{et}, and I_{dt}, i.e., $I_t = \{X_{1:t-1}, U_{1:t-1}, \tilde{X}_{1:t-1}, Y_{1:t-1}, S_{1:t-1}, \tilde{Y}_{1:t-1}\}$.

Proof. It is easy to see the validity of the following sequence of equalities:

$$\inf_{\mathbf{f},\mathbf{g},\mathbf{h}} J(\mathbf{f}, \mathbf{g}, \mathbf{h})$$
$$= \inf_{\mathbf{f},\mathbf{g},\mathbf{h}} \mathbb{E}\left\{\sum_{t=1}^{T} cU_t + (X_t - \hat{X}_t)^2\right\}$$
$$= \inf_{f_1,g_1,h_1} \mathbb{E}\left\{cU_1 + (X_1 - \hat{X}_1)^2\right.$$

$$+\inf_{\mathbf{f}_{2:T},\mathbf{g}_{2:T},\mathbf{h}_{2:T}} \mathbb{E}\left\{\sum_{t=2}^{T} cU_t + (X_t - \hat{X}_t)^2\right\}\Bigg\}$$

$$= \inf_{f_1,g_1,h_1} \mathbb{E}\Big\{cU_1 + (X_1 - \hat{X}_1)^2 + \inf_{f_2,g_2,h_2} \mathbb{E}\Big\{cU_2 + (X_2 - \hat{X}_2)^2 + \cdots + \inf_{f_T,g_T,h_T} \mathbb{E}\big\{cU_T + (X_T - \hat{X}_T)^2\big\}\dots\Big\}\Big\}.$$

Then, at time $t = T$, the optimization problem is to design (f_T, g_T, h_T) minimizing

$$J_1(f_T, g_T, h_T) := \mathbb{E}\left\{cU_T + (X_T - \hat{X}_T)^2\right\},$$

call it *Problem 1*. Recall that the decisions at time T are generated by $U_T = f_T(X_T, I_{sT})$, $Y_T = g_T(\tilde{X}_T, I_{eT})$, $\hat{X}_T = h_T(\tilde{Y}_T, S_T, I_{dT})$. We will show that using information about the past (I_{sT}, I_{eT}, I_{dT}) when making decisions cannot help improve the performance (that is, reduce the expected cost). Consider another problem, call it *Problem 2*, where I_T is available to all the decision makers, and one needs to design (f'_T, g'_T, h'_T) minimizing

$$J_2(f'_T, g'_T, h'_T) := \mathbb{E}\left\{cU_T + (X_T - \hat{X}_T)^2\right\},$$

where $U_T = f'_T(X_T, I_T)$, $Y_T = g'_T(\tilde{X}_T, I_T)$, $\hat{X}_T = h'_T(\tilde{Y}_T, S_T, I_T)$. Since the sensor, the encoder, and the decoder can always ignore the redundant information and behave as if they only know I_{sT}, I_{eT}, I_{dT}, respectively, the optimal cost in *Problem 2* is upper bounded by that in *Problem 1*, i.e.,

$$\inf_{(f'_T,g'_T,h'_T)} J_2(f'_T, g'_T, h'_T) \leq \inf_{(f_T,g_T,h_T)} J_1(f_T, g_T, h_T).$$

Similarly, consider a third problem, call it *Problem 3*, where I_{sT}, I_{eT}, I_{dT} are not available to the sensor, the encoder, and the decoder, respectively. One needs to design (f''_T, g''_T, h''_T) to minimize

$$J_3(f''_T, g''_T, h''_T) := \mathbb{E}\left\{cU_T + (X_T - \hat{X}_T)^2\right\},$$

where $U_T = f''_T(X_T)$, $Y_T = g''_T(\tilde{X}_T)$, $\hat{X}_T = h''_T(\tilde{Y}_T, S_T)$. By a similar argument as above, the system in *Problem 1* cannot perform worse than the system in *Problem 3*. Hence,

$$\inf_{(f_T,g_T,h_T)} J_1(f_T, g_T, h_T) \leq \inf_{(f''_T,g''_T,h''_T)} J_3(f''_T, g''_T, h''_T).$$

Let us come back to *Problem 2*. One can observe that the communication cost c, the distortion function $\rho(\cdot,\cdot)$, and the power constraint of the encoder do not depend on I_T. Furthermore, since $\{X_t\}$ and $\{V_t\}$ are i.i.d. stochastic processes, X_T and V_T are also independent of I_T. Therefore, there is no loss of optimality in restricting $U_T = f'_T(X_T)$, $Y_T = g'_T(\tilde{X}_T)$, $\hat{X}_T = h'_T(\tilde{Y}_T, S_T)$, and thus

$$\inf_{(f'_T,g'_T,h'_T)} J_2(f'_T, g'_T, h'_T) = \inf_{(f''_T,g''_T,h''_T)} J_3(f''_T, g''_T, h''_T).$$

The equality above indicates that in *Problem 1*, the sensor, the encoder, and the decoder can safely ignore their information about the past, namely I_{sT}, I_{eT}, and I_{dT}, when making decisions.

Since (f_T, g_T, h_T) do not take I_T as a parameter, the design of (f_T, g_T, h_T) is independent of the design of $(f_{1:T-1}, g_{1:T-1}, h_{1:T-1})$. Consequently, the problem can be viewed as a $(T-1)$-stage problem and a single-stage problem. By induction, we can show that $(f_1, g_1, h_1), (f_2, g_2, h_2), \dots, (f_T, g_T, h_T)$ can be designed independently, and (f_t, g_t, h_t) is designed to minimize the stage-wise cost $\mathbb{E}\{cU_t + (X_t - \hat{X}_t)^2\}$. Hence, the optimal decision policies (f_t, g_t, h_t) are in the form of (1). Furthermore, since $\{X_t\}$ and $\{V_t\}$ are i.i.d. stochastic processes, the optimal decision policies (f_t, g_t, h_t) should be the same for all $t = 1, 2, \dots, T$. □

By Theorem 1, the optimization problem with soft constraint can be reduced to a "single-stage" problem. Therefore, for simplicity we will suppress the subscript for time in all the expressions for the rest of this section. To present our main results for the single-stage problem, we need the following four assumptions.

Assumption 1. The source density p_X is nonatomic, even, and log-concave with support $\mathbb{R}$. Furthermore, p_X is continuously differentiable on $(0, \infty)$ (and on $(-\infty, 0)$ by symmetry).

Remark 2. There are several probability density functions satisfying Assumption 1, e.g., zero-mean Gaussian distribution, zero-mean Laplace distribution, and a few others. For simplicity, we assume here that p_X has support $\mathbb{R}$. However, the results also hold for the source density with support $(-a, a)$, $a > 0$, e.g., uniform distribution. In this case, we require that p_X is continuously differentiable on $(0, a)$.

Given any communication scheduling policy f, let $\mathcal{T}_0^f$, $\mathcal{T}_{1+}^f$, and $\mathcal{T}_{1-}^f$ be the *non-transmission region*, the *positive transmission region* and the *negative transmission region*, corresponding to f, where

$$\mathcal{T}_0^f := \{x \in \mathbb{R} | f(x) = 0\}, \quad \mathcal{T}_{1+}^f := \{x > 0 | f(x) = 1\},$$
$$\mathcal{T}_{1-}^f := \{x < 0 | f(x) = 1\}.$$

Note that $\mathcal{T}_0^f$, $\mathcal{T}_{1+}^f$, and $\mathcal{T}_{1-}^f$ may not be connected regions. Then, we make the following assumption on the communication scheduling policy.

Assumption 2. The sensor is restricted to apply the communication scheduling policy f satisfying

$$\mathbb{E}[X|X \in \mathcal{T}_{1-}^f] < \mathbb{E}[X|X \in \mathcal{T}_0^f] < \mathbb{E}[X|X \in \mathcal{T}_{1+}^f]. \tag{2}$$

Remark 3. There is a wide class of communication scheduling policies satisfying inequality (2). For example, given any communication scheduling policy f symmetric about 0, i.e., $f(x) = f(-x) \in \{0, 1\}$, and any even source density function p_X, we have

$$\mathbb{E}[X|X \in \mathcal{T}_{1-}^f] < 0 = \mathbb{E}[X|X \in \mathcal{T}_0^f] < \mathbb{E}[X|X \in \mathcal{T}_{1+}^f].$$

Then, Assumption 2 is satisfied.

Assumption 3. The communication channel noise V has zero mean, and finite variance, denoted by σ_V^2.

Assumption 4. The encoder and the decoder are restricted to apply piecewise affine policies:

$$g(\tilde{X}) = \begin{cases} S\alpha(S)\left(X - \mathbb{E}[X|U=1,S]\right), & \text{if } U = 1 \\ 0, & \text{if } U = 0 \end{cases}$$

$$h(\tilde{Y}, S) = \begin{cases} S\dfrac{1}{\alpha(S)}\dfrac{\gamma}{\gamma+1}\tilde{Y} + \mathbb{E}[X|U=1,S], & \text{if } U = 1 \\ \mathbb{E}[X|U=0], & \text{if } U = 0 \end{cases}$$

where $\gamma = P_T/\sigma_V^2$ is the signal-to-noise ratio (SNR), $\alpha(S) = \sqrt{P_T/\mathrm{Var}(X|U=1,S)}$ is the amplifying ratio, and $\mathrm{Var}(X|U=1,S)$ is the conditional variance.

It can be checked that when applying the encoding policy described above, the power consumption of the encoder meets the average power constraint (more details can be found in Gao, Akyol, & Başar, 2016d). Moreover, the events $U = 0$, $(U = 1, S = -1)$, and $(U = 1, S = 1)$ are equivalent to the events $X \in \mathcal{T}_0^f$, $X \in \mathcal{T}_{1-}^f$, and $X \in \mathcal{T}_{1+}^f$, respectively. Therefore, the encoding and decoding policies (g, h) are induced by the source density p_X and the communication scheduling policy f. For simplicity, we use $J(f)$ instead of $J(f, g, h)$ to denote the cost functional in the rest of this section.

Remark 4. Note that the assumption of piece-wise affine encoding policies originates from a prior work (Akyol, Rose, & Başar, 2015), which analyzed a memoryless zero-sum jamming game between a pair of transmitter and receiver and an adversary that generates an additive channel noise subject to second order (power) statistical constraints. It was shown in Akyol et al. (2015) that the saddle-point equilibrium associated with this zero-sum game is achieved by affine encoding/decoding policies for the transmitter–receiver pair. Here, we utilize such piece-wise affine policies, not only because they facilitate a tractable analysis but also because they possess such min–max robustness properties (see Akyol et al., 2015 for more details).

Under Assumptions 1–4, the optimal communication scheduling policy turns out to be unique, threshold-based and symmetric around zero, as stated in Theorem 2.

Theorem 2. *Consider the single-stage problem under Assumptions 1–4. Then, the optimal communication scheduling policy is of the threshold type:*

$$f(x)=\begin{cases}0, & \text{if } |x|<\beta\\ 1, & \text{otherwise}\end{cases}$$

where $\beta>0$ is the threshold. Furthermore, there exists a unique value β^* minimizing the cost functional $J(f)$ among all such thresholds.

To prove Theorem 2, we need the following definitions and lemmas. We first introduce a quantization problem.

Quantization Problem: The problem is one of quantizing the realizations (denoted by x) of a real-valued random variable (denoted by X) to $\mathcal{N}$ codepoints ($\mathcal{N}$ is finite and is known) according to some quantization rule (or quantizer) Q, i.e,

$$Q(x)=q_i,\ \text{if } x\in S_i,\quad i\in\{1,2,\ldots,\mathcal{N}\},$$

where $S_1,S_2,\ldots,S_{\mathcal{N}}$ are called quantization regions and $q_1,q_2,\ldots,q_{\mathcal{N}}$ are the corresponding codepoints. Note that $S_1,S_2,\ldots,S_{\mathcal{N}}$ are mutually disjoint sets and their union equals $\mathbb{R}$. The distortion error between a realization x and the its quantized value $Q(x)$ is $\rho(|x-Q(x)|)$, where $\rho:[0,\infty)\to[0,\infty)$ is called the distortion function. The performance of the quantizer Q is evaluated by its mean distortion error, denoted by $D(Q)$, i.e.,

$$D(Q):=\mathbb{E}\big[\rho\big(|X-Q(X)|\big)\big].$$

Then, given the probability distribution of X, the optimization problem is to design a quantizer $Q=Q^*$ (i.e., design $\{S_1,S_2,\ldots,S_{\mathcal{N}}\}$ and $\{q_1,q_2,\ldots,q_{\mathcal{N}}\}$) that minimizes $D(Q)$.

We recall here a result on the regularity of the optimal quantizer, as described in Lemma 1. The lemma says that given any quantizer, we can build another quantizer achieving no larger mean distortion error, by rearranging the quantization regions. Furthermore, the rearranged quantization regions are connected and have the same probability measure with the original quantization regions.

Lemma 1 (*György & Linder, 2002, Theorem 1 and Corollary 1*). *Assume that the source X has nonatomic distribution p_X, and $\rho:[0,\infty)\to[0,\infty)$ is convex and non-decreasing. Then, for any $\mathcal{N}$-level quantizer Q with quantization regions $\{S_1,S_2,\ldots,S_{\mathcal{N}}\}$ and the corresponding codepoints $\{q_1,q_2,\ldots,q_{\mathcal{N}}\}$, there exists a quantizer $\hat{Q}$ with quantization regions $\{\hat{S}_1,\hat{S}_2,\ldots,\hat{S}_{\mathcal{N}}\}$ and the corresponding codepoints $\{\hat{q}_1,\hat{q}_2,\ldots,\hat{q}_{\mathcal{N}}\}$ such that*

(1) *$\hat{S}_i$ is convex, and $\mathbb{P}(X\in\hat{S}_i)=\mathbb{P}(X\in S_i)$, for all $i=1,\ldots,\mathcal{N}$.*
(2) *If $q_i<q_j$, then $\hat{S}_i<\hat{S}_j$, i.e., $x<y$ for any $x\in\hat{S}_i$ and $y\in\hat{S}_j$.*
(3) *$D(\hat{Q})\le D(Q)$.*

Now returning to our problem, for any communication scheduling policy f, we can construct a three-level quantizer, denoted by Q^f, with quantizing regions $(\mathcal{T}_0^f,\mathcal{T}_{1+}^f,\mathcal{T}_{1-}^f)$ and the corresponding codepoints $(\mathbb{E}[X|X\in\mathcal{T}_0^f],\mathbb{E}[X|X\in\mathcal{T}_{1+}^f],\mathbb{E}[X|X\in\mathcal{T}_{1-}^f])$. Let $D(Q^f)$ be the mean squared distortion of Q^f, i.e.,

$$\begin{aligned}&D(Q^f)\\&=\mathbb{E}\Big[\big(X-Q^f(X)\big)^2\Big]\\&=\sum_{i\in I}\mathbb{E}\Big[\big(X-\mathbb{E}[X|X\in\mathcal{T}_i^f]\big)^2|X\in\mathcal{T}_i^f\Big]\,\mathbb{P}(X\in\mathcal{T}_i^f)\\&=\sum_{i\in I}\mathrm{Var}(X|X\in\mathcal{T}_i^f)\,\mathbb{P}(X\in\mathcal{T}_i^f),\end{aligned}$$

where $I=\{0,1+,1-\}$. By Lemma 1, we have the following result.

Lemma 2. *Suppose the source density p_X is nonatomic and even. Then, for any communication scheduling policy f satisfying Assumption 2, we can construct a threshold-based communication scheduling policy $f^{(1)}$ such that*

(1) *$\mathcal{T}_0^{f^{(1)}}=(-\beta_2,\beta_1)$, $\mathcal{T}_{1+}^{f^{(1)}}=(\beta_1,\infty)$, and $\mathcal{T}_{1-}^{f^{(1)}}=(-\infty,-\beta_2)$, where $\beta_1,\beta_2>0$ are thresholds.*
(2) *$\mathbb{P}(X\in\mathcal{T}_i^{f^{(1)}})=\mathbb{P}(X\in\mathcal{T}_i^f)$, for all $i\in\{0,1+,1-\}$.*
(3) *$D(Q^{f^{(1)}})\le D(Q^f)$.*

Proof. By Lemma 1, given a three-level quantizer Q^f, there exists a three-level quantizer $\hat{Q}$ with quantization regions $(\hat{S}_0,\hat{S}_{1+},\hat{S}_{1-})$ and corresponding codepoints $(\hat{q}_0,\hat{q}_{1+},\hat{q}_{1-})$ such that

(1) $\hat{S}_i$ is convex, and $\mathbb{P}(X\in\hat{S}_i)=\mathbb{P}(X\in\mathcal{T}_i^f)$, for all $i\in\{0,1+,1-\}$.
(2) $\hat{S}_{1-}<\hat{S}_0<\hat{S}_{1+}$.
(3) $D(\hat{Q})\le D(Q)$.

The second item holds since $\mathbb{E}[X|X\in\mathcal{T}_{1-}^f]<\mathbb{E}[X|X\in\mathcal{T}_0^f]<\mathbb{E}[X|X\in\mathcal{T}_{1+}^f]$ (Assumption 2). Note that since $\mathcal{T}_{1+}^f\subseteq(0,\infty)$, $\mathcal{T}_{1-}^f\subseteq(-\infty,0)$, and the source density p_X is even, we have

$$\mathbb{P}(X\in\hat{S}_{1+})=\mathbb{P}(X\in\mathcal{T}_{1+}^f)\le\frac{1}{2},$$
$$\mathbb{P}(X\in\hat{S}_{1-})=\mathbb{P}(X\in\mathcal{T}_{1-}^f)\le\frac{1}{2}.$$

Combining the above inequalities with the second item, we have $\hat{S}_{1+}=(\beta_1,\infty)$, $\hat{S}_{1-}=(-\infty,-\beta_2)$, and $\hat{S}_0=(-\beta_2,\beta_1)$ for some $\beta_1,\beta_2\ge0$. We now construct a threshold-based communication scheduling policy $f^{(1)}$ by letting $\mathcal{T}_i^{f^{(1)}}=\hat{S}_i$, $i\in\{0,1+,1-\}$. Since the distortion function is the squared error, the optimal codepoints corresponding to quantization regions $(\mathcal{T}_0^{f^{(1)}},\mathcal{T}_{1+}^{f^{(1)}},\mathcal{T}_{1-}^{f^{(1)}})$ are $(\mathbb{E}[X|X\in\mathcal{T}_0^{f^{(1)}}],\mathbb{E}[X|X\in\mathcal{T}_{1+}^{f^{(1)}}],\mathbb{E}[X|X\in\mathcal{T}_{1-}^{f^{(1)}}])$. Hence, we have $D(Q^{f^{(1)}})\le D(\hat{Q})\le D(Q^f)$. □

Note that $f^{(1)}$ constructed in Lemma 2 may or may not be symmetric around zero. We now have the following proposition, which states that based on $f^{(1)}$, we can further construct a threshold-based policy $f^{(2)}$, which is symmetric around zero and has no larger mean squared distortion. Furthermore, the probability measure over the non-transmission region of $f^{(2)}$ is the same as that of $f^{(1)}$.

Proposition 1. *Suppose the source density p_X satisfies Assumption 1. Then, for any communication scheduling policy f satisfying Assumption 2, we can construct a threshold-based communication scheduling policy $f^{(2)}$ symmetric around zero such that*

(1) *$\mathcal{T}_0^{f^{(2)}}=(-\beta,\beta)$, $\mathcal{T}_{1+}^{f^{(2)}}=(\beta,\infty)$, $\mathcal{T}_{1-}^{f^{(2)}}=(-\infty,-\beta)$, where $\beta>0$.*

(2) $\mathbb{P}(X \in \mathcal{T}_0^{f^{(2)}}) = \mathbb{P}(X \in \mathcal{T}_0^{f})$.
(3) $D(Q^{f^{(2)}}) \leq D(Q^f)$.

To prove Proposition 1, we need to apply results from majorization theory, introduced below. Given a Borel measurable set A, we use A^σ to denote its symmetric rearrangement, i.e., $A^\sigma = [-a, a]$, and $\mathcal{L}(A^\sigma) = \mathcal{L}(A)$ (same Lebesgue measure). Given a non-negative integrable function $p : \mathbb{R} \to \mathbb{R}$, we use p^σ to denote its symmetric rearrangement, which is described as follows:

$$p^\sigma(x) := \int_0^\infty \mathbf{1}_{\{z\in\mathbb{R}|p(z)\geq\rho\}^\sigma}(x)d\rho, \quad x \in \mathbb{R}.$$

$\mathbf{1}_{\{z\in\mathbb{R}|p(z)\geq\rho\}^\sigma}(x)$ is the indicator function on whether x is an element of $\{z \in \mathbb{R}|p(z) \geq \rho\}^\sigma$ or not, i.e.,

$$\mathbf{1}_{\{z\in\mathbb{R}|p(z)\geq\rho\}^\sigma}(x) = \begin{cases} 1, & \text{if } x \in \{z \in \mathbb{R}|p(z) \geq \rho\}^\sigma \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1. Given two probability densities p and q defined on $\mathbb{R}$, we say p majorizes q, denoted by $p \succ q$, if

$$\int_{|x|<t} q^\sigma(x)dx \leq \int_{|x|<t} p^\sigma(x)dx, \quad \text{for all } t \geq 0.$$

Lemma 3 (*Lipsa & Martins, 2011, Lemma 4*)**.** *Let p_X and $p_{X'}$ be probability density functions defined on $\mathbb{R}$. Assume that p_X is even and log-concave, and $p_X \succ p_{X'}$. Then,*

$$\int_{-\infty}^{\infty} x^2 p_X(x)dx \leq \int_{-\infty}^{\infty} (x-y)^2 p_{X'}(x)dx, \text{ for all } y \in \mathbb{R},$$

or equivalently, $\mathrm{Var}(X) \leq \mathrm{Var}(X')$.

Remark 5. Lemma 4 in Lipsa and Martins (2011) assumes that p_X is even, quasi-concave, and there exists $b \in \mathbb{R}$ such that p_X is non-decreasing on $(-\infty, b]$ and non-increasing on (b, ∞). Note that a positive log-concave function is also quasi-concave. Moreover, it can be shown that if p_X is even and log-concave, then p_X is non-decreasing on $(-\infty, 0]$ and non-increasing on $(0, \infty)$. In view of this, Lemma 3 is a slightly modified version of Lemma 4 in Lipsa and Martins (2011).

Lemma 4 (*Lipsa & Martins, 2011, Lemma 2*)**.** *Let p_X and $p_{X'}$ be probability density functions defined on $\mathbb{R}$. Assume that p_X is even and log-concave, and $p_X \succ p_{X'}$. Let $A = [-\tau, \tau]$ be any symmetric closed interval such that $\int_A p_X(x)dx > 0$ and let $h : \mathbb{R} \to [0, 1]$ be any function such that $\int_{\mathbb{R}} h(x)p_{X'}(x)dx = \int_A p_X(x)dx$. Then,*

$$p_{X|X\in A} \succ \frac{h \cdot p_{X'}}{\int_{\mathbb{R}} h(x)p_{X'}(x)dx}.$$

Combining Lemmas 3 and 4, we have the following lemma.

Lemma 5. *Let p_X be an even and log-concave density. Let $A = [-\tau, \tau]$ be any symmetric closed interval such that $\int_A p_X(x)dx > 0$, and let B be any subset of $\mathbb{R}$ such that $\int_B p_X(x)dx = \int_A p_X(x)dx$. Then,*

$\mathrm{Var}(X|X \in A) \leq \mathrm{Var}(X|X \in B)$.

Proof. One can see that p_X majorizes itself. Furthermore, we choose $h(x)$ to be the indicator function on whether x belongs to B or not, i.e., $h(x) = \mathbf{1}_{\{x\in B\}}$. Then, $\int_{\mathbb{R}} h(x)p_X(x)dx = \int_B p_X(x)dx = \int_A p_X(x)dx$. By Lemma 4, the conditional density $p_{X|X\in A}$ majorizes the conditional density $p_{X|X\in B}$. Since A is symmetric about zero, and p_X is even and log-concave, we have $p_{X|X\in A}$ is also even and log-concave. By Lemma 3, we conclude that $\mathrm{Var}(X|X \in A) \leq \mathrm{Var}(X|X \in B)$. □

To prove Proposition 1, we also need to apply property of log-concave distribution, which is introduced below.

Lemma 6 (*Bagnoli & Bergstrom, 2005, Theorem 6*)**.** *Let p_X be a continuously differentiable and log-concave probability density function defined on (a, b). Let β be a variable belonging to interval (a, b). Then, the function $G_X(\beta)$, defined below, is monotone decreasing in β:*

$$G_X(\beta) := \mathbb{E}[X|X > \beta] - \beta. \tag{3}$$

Note that a and b in Lemma 6 can be $-\infty$ and ∞, respectively. We will frequently refer to this function $G_X(\beta)$[1] in the rest of the paper. We next provide an extension of Lemma 6 as follows.

Lemma 7. *Let p_X be an even and log-concave probability density function defined on $\mathbb{R}$. Furthermore, let p_X be continuously differentiable on $(0, \infty)$ and $(-\infty, 0)$, and let β taking values in $(0, \infty)$. Then, $G_X(\beta)$ as defined by (3) is monotone decreasing in β for $\beta \in (0, \infty)$.*

Proof. Let Y be a random variable such that $Y = |X|$. Denote by p_Y the probability function of Y. Since the probability density of X, p_X is even, we have

$$p_Y(y) = \begin{cases} 2p_X(y), & \text{if } y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since p_X is continuously differentiable on $(0, \infty)$, so is p_Y. Furthermore, it can be shown quite readily that for any $\beta \in (0, \infty)$, $\mathbb{E}[Y|Y > \beta] = \mathbb{E}[X|X > \beta]$. Then, we have $G_Y(\beta) = G_X(\beta)$. By Lemma 6, $G_Y(\beta)$ is monotone decreasing in β. Hence, we conclude that $G_X(\beta)$ is also monotone decreasing in β. □

We are now in a position to prove Proposition 1.

Proof of Proposition 1. By Lemma 2, given any communication scheduling policy $f^{(0)}$ satisfying Assumption 2, we can construct a threshold-based policy $f^{(1)}$ such that

(1) $\mathcal{T}_0^{f^{(1)}} = (-\beta_2, \beta_1)$, $\mathcal{T}_{1+}^{f^{(1)}} = (\beta_1, \infty)$, and $\mathcal{T}_{1-}^{f^{(1)}} = (-\infty, -\beta_2)$.
(2) $\mathbb{P}(X \in \mathcal{T}_i^{f^{(1)}}) = \mathbb{P}(X \in \mathcal{T}_i^{f^{(0)}})$, for all $i \in \{0, 1+, 1-\}$.
(3) $D(Q^{f^{(1)}}) \leq D(Q^{f^{(0)}})$.

Based on policy $f^{(1)}$, we can construct a threshold-based policy $f^{(2)}$ symmetric around zero such that

(1) $\mathcal{T}_0^{f^{(2)}} = (-\beta, \beta)$, $\mathcal{T}_{1+}^{f^{(2)}} = (\beta, \infty)$, $\mathcal{T}_{1-}^{f^{(2)}} = (-\infty, -\beta)$.
(2) $\mathbb{P}(X \in \mathcal{T}_0^{f^{(2)}}) = \mathbb{P}(X \in \mathcal{T}_0^{f^{(1)}})$.

Then, we only need to show that $D(Q^{f^{(2)}}) \leq D(Q^{f^{(1)}})$. Note that $D(Q^{f^{(1)}})$ and $D(Q^{f^{(2)}})$ can be expressed as

$$D(Q^{f^{(1)}}) = \sum_{i\in I} \mathrm{Var}(X|X \in \mathcal{T}_i^{f^{(1)}})\, \mathbb{P}(X \in \mathcal{T}_i^{f^{(1)}})$$
$$D(Q^{f^{(2)}}) = \sum_{i\in I} \mathrm{Var}(X|X \in \mathcal{T}_i^{f^{(2)}})\, \mathbb{P}(X \in \mathcal{T}_i^{f^{(2)}})$$

where $I = \{0, 1+, 1-\}$. By Lemma 5, we obtain $\mathrm{Var}(X|X \in \mathcal{T}_0^{f^{(2)}}) \leq \mathrm{Var}(X|X \in \mathcal{T}_0^{f^{(1)}})$. Since $\mathbb{P}(X \in \mathcal{T}_0^{f^{(2)}}) = \mathbb{P}(X \in \mathcal{T}_0^{f^{(1)}})$, we have $\mathrm{Var}(X|X \in \mathcal{T}_0^{f^{(2)}})\mathbb{P}(X \in \mathcal{T}_0^{f^{(2)}}) \leq \mathrm{Var}(X|X \in \mathcal{T}_0^{f^{(1)}})\mathbb{P}(X \in \mathcal{T}_0^{f^{(1)}})$. Hence, we will be done if we show that

$$\sum_{i\in\{1+,1-\}} \mathrm{Var}(X|X \in \mathcal{T}_i^{f^{(2)}})\, \mathbb{P}(X \in \mathcal{T}_i^{f^{(2)}}) \leq \sum_{i\in\{1+,1-\}} \mathrm{Var}(X|X \in \mathcal{T}_i^{f^{(1)}})\, \mathbb{P}(X \in \mathcal{T}_i^{f^{(1)}}).$$

Consider the class of threshold-based communication scheduling policies, denoted by $\mathcal{F}$, whose generic element f is in the form of

$$\mathcal{T}_0^f = (-\eta_2, \eta_1), \quad \mathcal{T}_{1+}^f = (\eta_1, \infty), \quad \mathcal{T}_{1-}^f = (-\infty, -\eta_2),$$

[1] $G_X(\beta)$ is also called the *mean residual lifetime*.

where $\eta_1, \eta_2 \geq 0$, and

$$\mathbb{P}(X \in \mathcal{T}_0^f) = \mathbb{P}(X \in \mathcal{T}_0^{f^{(0)}}) = k.$$

It is clear that $f^{(1)}$ and $f^{(2)}$ are elements of $\mathcal{F}$. Let $PD(Q^f)$ be the sum of the mean squared distortions of Q^f over regions $\mathcal{T}_{1+}^f$ and $\mathcal{T}_{1-}^f$, i.e.,

$$\begin{aligned} PD(Q^f) &:= \sum_{i \in \{1+,1-\}} \mathrm{Var}(X|X \in \mathcal{T}_i^f)\, \mathbb{P}(X \in \mathcal{T}_i^f) \\ &= \mathrm{Var}\big(X|X < -\eta_2\big)\, \mathbb{P}\big(X < -\eta_2\big) \\ &\quad + \mathrm{Var}\big(X|X > \eta_1\big)\, \mathbb{P}\big(X > \eta_1\big) \\ &= \mathrm{Var}\big(X|X > \eta_2\big)\, \mathbb{P}\big(X > \eta_2\big) \\ &\quad + \mathrm{Var}\big(X|X > \eta_1\big)\, \mathbb{P}\big(X > \eta_1\big), \end{aligned}$$

where the last equality holds since p_X is even. We now show that $f^{(2)}$ is the global minimizer of $PD(Q^f)$ among all elements in $\mathcal{F}$. Since $\mathbb{P}(X \in \mathcal{T}_0^f) = k$, we have

$$\int_{-\infty}^{-\eta_2} p_X(x)dx + \int_{\eta_1}^{\infty} p_X(x)dx = 1 - k.$$

Taking the derivatives of both sides with respect to η_1, we have

$$\frac{d\eta_2}{d\eta_1} \cdot \frac{\partial}{\partial \eta_2} \int_{-a}^{-\eta_2} p_X(x)dx + \frac{\partial}{\partial \eta_1} \int_{\eta_1}^{a} p_X(x)dx = 0,$$

which implies that

$$\frac{d\eta_2}{d\eta_1} = -\frac{p_X(\eta_1)}{p_X(-\eta_2)} = -\frac{p_X(\eta_1)}{p_X(\eta_2)}. \tag{i}$$

The equality above holds because p_X is even. Now taking the derivative of $PD(Q^f)$ with respect to η_1, we have

$$\begin{aligned} \frac{d}{d\eta_1} PD(Q^f) &= \frac{d\eta_2}{d\eta_1} \cdot \frac{\partial}{\partial \eta_2} \mathrm{Var}\big(X|X > \eta_2\big)\, \mathbb{P}\big(X > \eta_2\big) \\ &\quad + \frac{\partial}{\partial \eta_1} \mathrm{Var}\big(X|X > \eta_1\big)\, \mathbb{P}\big(X > \eta_1\big). \end{aligned} \tag{ii}$$

The second term in (ii) can be computed as (details of this derivation can be found in Gao et al., 2016d):

$$\begin{aligned} &\frac{\partial}{\partial \eta_1} \mathrm{Var}(X|X > \eta_1)\, \mathbb{P}(X > \eta_1) \\ &= -p_X(\eta_1) \cdot \big(\eta_1 - \mathbb{E}[X|X > \eta_1]\big)^2. \end{aligned} \tag{iii}$$

Similarly, we can simplify the first term in (ii) to

$$\begin{aligned} &\frac{\partial}{\partial \eta_2} \mathrm{Var}\big(X|X > \eta_2\big)\, \mathbb{P}\big(X > \eta_2\big) \\ &= -p_X(\eta_2) \cdot \big(\eta_2 - \mathbb{E}[X|X > \eta_2]\big)^2. \end{aligned} \tag{iv}$$

Plugging (i), (iii), and (iv) into (ii), we have

$$\begin{aligned} &\frac{d}{d\eta_1} PD(Q^f) \\ &= -\frac{p_X(\eta_1)}{p_X(\eta_2)} \cdot -p_X(\eta_2) \cdot \big(\eta_2 - \mathbb{E}[X|X > \eta_2]\big)^2 \\ &\quad - p_X(\eta_1) \cdot \big(\eta_1 - \mathbb{E}[X|X > \eta_1]\big)^2 \\ &= p_X(\eta_1)\Big[\big(\eta_2 - \mathbb{E}[X|X > \eta_2]\big)^2 - \big(\eta_1 - \mathbb{E}[X|X > \eta_1]\big)^2\Big] \\ &= p_X(\eta_1)\big(G_X^2(\eta_2) - G_X^2(\eta_1)\big). \end{aligned}$$

By Lemma 7, $G_X(\eta)$ is a non-negative and monotone decreasing function. Therefore, $G_X^2(\eta)$ is monotone decreasing, and $dPD(Q^f)/d\eta_1 \geq 0$ if $\eta_1 > \eta_2$, $dPD(Q^f)/d\eta_1 = 0$ if $\eta_1 = \eta_2$, and $dPD(Q^f)/d\eta_1 \leq 0$ if $\eta_1 < \eta_2$. This implies that $\eta_1 = \eta_2$ (corresponding to $f^{(2)}$) is a global minimizer. □

Now we are in a position to prove Theorem 2 by applying Proposition 1. The approach of the proof can be summarized as follows: (1) Given any communication scheduling policy f, it can be computed that the cost functional $J(f)$ consists of two parts: the first part is the mean squared distortion of Q^f, and the second part is a cost functional in the noiseless-channel setting. (2) Based on f, we can construct a symmetric threshold-based communication scheduling policy f', which has the same probability measure on the non-transmission region. (3) By Proposition 1, if f satisfies Assumption 2, then the first part in the cost functional of f' is lower than that of f. By Lemma 5, if the source density is even and log-concave (which is also unimodal), then the second part in the cost functional of f' is also lower than that of f. (4) Without loss of optimality, we can consider only the class of threshold-based policies symmetric around zero. By Lemma 7, it can be shown that there exists a unique optimal threshold minimizing the cost functional. We now proceed with the details of the proof of the theorem.

Proof of Theorem 2. Consider any communication scheduling policy f. The expected cost corresponding to f can be computed as follows:

$$\begin{aligned} J(f) &= \mathbb{E}\Big[cU + (X - \hat{X})^2\Big] \\ &= \sum_{i \in I} \mathbb{E}\Big[cU + (X - \hat{X})^2 | X \in \mathcal{T}_i^f\Big]\ \mathbb{P}(X \in \mathcal{T}_i^f), \end{aligned}$$

where $I = \{0, 1+, 1-\}$. When $X \in \mathcal{T}_0^f$, we have $U = 0$ and $\hat{X} = \mathbb{E}[X|X \in \mathcal{T}_0^f]$. Hence,

$$\begin{aligned} &\mathbb{E}\Big[cU + (X - \hat{X})^2 | X \in \mathcal{T}_0^f\Big] \\ &= \mathbb{E}\Big[\big(X - \mathbb{E}\big[X|X \in \mathcal{T}_0^f\big]\big)^2 | X \in \mathcal{T}_0^f\Big] \\ &= \mathrm{Var}(X|X \in \mathcal{T}_0^f). \end{aligned}$$

When $X \in \mathcal{T}_{1+}^f$, we have $U = 1$, and $Y = \alpha(1)\big(X - \mathbb{E}\big[X|X \in \mathcal{T}_{1+}^f\big]\big)$. Hence,

$$\begin{aligned} \hat{X} &= \frac{1}{\alpha(1)} \frac{\gamma}{\gamma+1} \tilde{Y} + \mathbb{E}\big[X|X \in \mathcal{T}_{1+}^f\big] \\ &= \frac{\gamma}{\gamma+1} X + \frac{1}{\alpha(1)} \frac{\gamma}{\gamma+1} V + \frac{1}{\gamma+1} \mathbb{E}\big[X|X \in \mathcal{T}_{1+}^f\big], \end{aligned}$$

where $\alpha(1) = \sqrt{P_T/\mathrm{Var}(X|X \in \mathcal{T}_{1+}^f)}$, and $\gamma = P_T/\sigma_V^2$. Hence, it can be shown that (for details of the derivation, see Gao et al., 2016d)

$$\mathbb{E}\big[cU + (X - \hat{X})^2 | X \in \mathcal{T}_{1+}^f\big] = c + \frac{1}{\gamma+1} \mathrm{Var}(X|X \in \mathcal{T}_{1+}^f).$$

Similarly, one can compute that

$$\mathbb{E}\big[cU + (X - \hat{X})^2 | X \in \mathcal{T}_{1-}^f\big] = c + \frac{1}{\gamma+1} \mathrm{Var}(X|X \in \mathcal{T}_{1-}^f).$$

Hence, $J(f)$ can be further expressed as

$$\begin{aligned} J(f) &= \mathrm{Var}(X|X \in \mathcal{T}_0^f)\mathbb{P}(X \in \mathcal{T}_0^f) \\ &+ \frac{1}{\gamma+1} \mathrm{Var}(X|X \in \mathcal{T}_{1+}^f)\mathbb{P}(X \in \mathcal{T}_{1+}^f) + c\ \mathbb{P}(X \in \mathcal{T}_{1+}^f) \\ &+ \frac{1}{\gamma+1} \mathrm{Var}(X|X \in \mathcal{T}_{1-}^f)\mathbb{P}(X \in \mathcal{T}_{1-}^f) + c\ \mathbb{P}(X \in \mathcal{T}_{1-}^f) \\ &= \frac{1}{\gamma+1} D(Q^f) + \frac{\gamma}{\gamma+1} \mathrm{Var}(X|X \in \mathcal{T}_0^f)\mathbb{P}(X \in \mathcal{T}_0^f) \\ &\quad + c\ \mathbb{P}(X \in \mathcal{T}_{1+}^f) + c\ \mathbb{P}(X \in \mathcal{T}_{1-}^f). \end{aligned} \tag{v}$$

Given any communication scheduling policy f, we can construct a threshold-based communication scheduling policy f' symmetric around zero such that

(1) $\mathcal{T}_0^{f'} = (-\beta, \beta)$, $\mathcal{T}_{1+}^{f'} = (\beta, \infty)$, $\mathcal{T}_{1-}^{f'} = (-\infty, -\beta)$.
(2) $\mathbb{P}(X \in \mathcal{T}_0^{f'}) = \mathbb{P}(X \in \mathcal{T}_0^f)$, or equivalently, $\mathbb{P}(X \in \mathcal{T}_{1+}^{f'}) + \mathbb{P}(X \in \mathcal{T}_{1-}^{f'}) = \mathbb{P}(X \in \mathcal{T}_{1+}^f) + \mathbb{P}(X \in \mathcal{T}_{1-}^f)$.

By Proposition 1 and Lemma 5, we have $D(Q^{f'}) \leq D(Q^f)$ and $\mathrm{Var}(X|X \in \mathcal{T}_0^{f'}) \leq \mathrm{Var}(X|X \in \mathcal{T}_0^f)$. Furthermore, we have $\mathbb{P}(X \in \mathcal{T}_0^{f'}) = \mathbb{P}(X \in \mathcal{T}_0^f)$ and $c\ \mathbb{P}(X \in \mathcal{T}_{1+}^{f'}) + c\ \mathbb{P}(X \in \mathcal{T}_{1-}^{f'}) = c\ \mathbb{P}(X \in \mathcal{T}_{1+}^f) + c\ \mathbb{P}(X \in \mathcal{T}_{1-}^f)$. Hence, we conclude that $J(f') \leq J(f)$, which implies that without loss of optimality.

The result above implies that without loss of optimality, we can restrict the search of the optimal communication scheduling policy to the class of symmetric threshold-based type. Denote by $J(\beta)$ the expected cost corresponding to a symmetric threshold-based communication scheduling policy with threshold β, where $\beta \geq 0$. By (v), $J(\beta)$ can be computed as

$$\begin{aligned} J(\beta) &= \int_{-\beta}^{\beta} x^2 p_X(x)dx + \frac{1}{\gamma+1}\mathrm{Var}(X|X < -\beta) \\ &\quad \cdot \mathbb{P}(X < -\beta) + c\int_{-\infty}^{-\beta} p_X(x)dx + \frac{1}{\gamma+1} \\ &\quad \cdot \mathrm{Var}(X|X > \beta)\mathbb{P}(X > \beta) + c\int_{\beta}^{\infty} p_X(x)dx \\ &= 2\int_0^{\beta} x^2 p_X(x)dx + 2\frac{1}{\gamma+1}\mathrm{Var}(X|X > \beta) \\ &\quad \cdot \mathbb{P}(X > \beta) + 2c\int_{\beta}^{\infty} p_X(x)dx, \end{aligned}$$

where the second equality holds since p_X is even. Taking the derivative of $J(\beta)$ with respect to β, and by (iii), we have

$$\begin{aligned} \frac{d}{d\beta}J(\beta) &= 2p_X(\beta)\big(\beta^2 - \frac{1}{\gamma+1}\big(\mathbb{E}[X|X > \beta] - \beta\big)^2 - c\big) \\ &= 2p_X(\beta)\big(\beta^2 - \frac{1}{\gamma+1}G_X^2(\beta) - c\big). \end{aligned}$$

Since $c > 0$ and $G_X(\beta)$ is monotone decreasing, there exists a unique β^* in $[0, \infty)$ such that

$$\beta^{*2} = \frac{1}{\gamma+1}G_X^2(\beta^*) + c.$$

Furthermore, $dJ(\beta)/d\beta < 0$ when $\beta < \beta^*$ and $dJ(\beta)/d\beta > 0$ when $\beta > \beta^*$. Hence, β^* is the unique global minimizer among all $\beta \geq 0$. □

Remark 6. If the density function p_X has support $(-a, a)$ and $0 < a < \beta^*$, then $dJ(\beta)/d\beta$ is always negative, which implies that the minimizing β is just a. This means that the optimal communication scheduling policy is to always choose no transmission regardless of sensor's observation. Such a case can occur when the communication cost is very high.

4. Optimization problem with hard constraint

To present our main results for the problem with the hard constraint, we introduce a number of terms. First, we let E_t denote the number of remaining communication opportunities at the beginning of the tth time interval, i.e., $E_t = N - \sum_{i=1}^{t-1} U_i$. Then, evolution of E_t is described by

$$E_t = E_{t-1} - U_{t-1}, t \geq 2; \quad E_1 = N\,. \tag{4}$$

Furthermore, the communication constraint is

$$U_t \leq E_t, \quad \text{for all } t = 1, 2, \ldots, T. \tag{5}$$

Recall that $U_{1:t-1}$ is the common information shared by all the decision makers, and hence E_t is also known by all the decision makers.

Second, we let $J^*(t, E_t)$ be the optimal cost-to-go if the system is initialized at time t (or equivalently, at the beginning of the tth time interval) with E_t number of communication opportunities. Specifically, we define $J^*(T+1, \cdot) = 0$ for any number of communication opportunities.

Third, for any $E_t > 0$, we let $c(t, E_t)$ denote the difference between two optimal cost-to-go, i.e.,

$$c(t, E_t) = J^*(t+1, E_t - 1) - J^*(t+1, E_t).$$

Remark 7. $c(t, E_t)$ can be interpreted as the opportunity cost for choosing to communicate with the decoder rather than not to communicate.

The following theorem ensures that without loss of optimality, we can restrict all the decision makers to consider only their current inputs and E_t when making decisions at time t. Furthermore, the optimal cost-to-go can be obtained via solving the dynamic programming equation.

Theorem 3. *Consider the optimization problem with hard constraint as formulated in Section 2.4. Without loss of optimality, we can restrict communication scheduling, encoding and decoding policies to the forms:*

$$U_t = f_t(X_t, E_t),\ Y_t = g_t(\tilde{X}_t, E_t),\ \hat{X}_t = h_t(\tilde{Y}_t, S_t, E_t).$$

Furthermore, the optimal cost-to-go $J^*(t, E_t)$ can be obtained by solving the dynamic programming (DP) equation:

$$\begin{aligned} J^*(T+1, \cdot) &= 0 \\ J^*(t, E_t) &= \inf_{f_t, g_t, h_t} \mathbb{E}\left\{(X_t - \hat{X}_t)^2 + J^*(t+1, E_{t+1})\right\}. \end{aligned} \tag{6}$$

The proof of Theorem 3 is similar to that of Theorem 1, and hence is not included here; it can be found in Gao et al. (2016d).

Regarding the DP equation (6), we have the following observations: (1) When $E_t = 0$, the sensor has no opportunity to make a transmission, and thus $U_t = 0$ regardless of the realization of X_t. In this scenario, the sensor's decision U_t does not contain any hidden information about X_t. Therefore, the optimal estimator is simply $\mathbb{E}[X_t]$, and the mean squared error is $\mathrm{Var}(X_t)$. Then, the DP equation can be easily updated as follows:

$$J^*(t, 0) = \mathrm{Var}(X_t) + J^*(t+1, 0).$$

(2) When $E_t > 0$, the DP equation can be written as

$$\begin{aligned} J^*(t, E_t) &= \inf_{f_t, g_t, h_t} \mathbb{E}\left\{(X_t - \hat{X}_t)^2 + J^*(t+1, E_{t+1})\right\} \\ &= J^*(t+1, E_t) + \inf_{f_t, g_t, h_t} \mathbb{E}\left\{c(t, E_t)U_t + (X_t - \hat{X}_t)^2\right\}. \end{aligned} \tag{7}$$

Note that the minimization in the second line of (7) is just the single-stage problem discussed in Section 3 with communication cost $c(t, E_t)$. This now motivates us to make the following two assumptions.

Assumption 5. The sensor is restricted to apply the communication scheduling policies f_t such that for any $1 \leq t \leq T$ and $E_t > 0$,

$$\begin{aligned} &\mathbb{E}[X_t|U_t = 1, E_t, X_t < 0] < \mathbb{E}[X_t|U_t = 0, E_t] \\ &< \mathbb{E}[X_t|U_t = 1, E_t, X_t > 0]. \end{aligned}$$

Assumption 6. The encoder and the decoder are restricted to apply piecewise affine policies:

$$g_t(\tilde{X}_t, E_t) = \begin{cases} S_t\alpha(S_t)\left(X_t - \mathbb{E}\left[X_t|U_t = 1, E_t, S_t\right]\right), & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0 \end{cases}$$

$$h_t(\tilde{Y}_t, S_t, E_t) = \begin{cases} S_t\frac{1}{\alpha(S_t)}\frac{\gamma}{\gamma+1}\tilde{Y}_t + \mathbb{E}\left[X_t|U_t = 1, E_t, S_t\right], & \text{if } U_t = 1 \\ \mathbb{E}[X_t|U_t = 0, E_t], & \text{if } U_t = 0 \end{cases}$$

where $\gamma = P_T/\sigma_V^2$, $\alpha(S_t) = \sqrt{P_T/\mathrm{Var}(X_t|U_t = 1, E_t, S_t)}$, and $\mathrm{Var}(X_t|U_t = 1, E_t, S_t)$.

Then, we have the following theorem on the optimality of symmetric threshold-based communication scheduling strategy. Its proof involves simply an application of Theorem 2, and hence is not included here.

Theorem 4. *Consider the problem with hard constraint under Assumption 1, 3, 5 and 6, the optimal communication scheduling policy f_t^* for the sensor is*

$$f_t^*(X_t, E_t) = \begin{cases} 1, & \text{if } E_t > 0 \text{ and } |X_t| > \beta_t^*(E_t) \\ 0, & \text{if } E_t = 0 \text{ or } \; |X_t| \le \beta_t^*(E_t) \end{cases} \quad (8)$$

where $\beta_t^*(E_t)$ is non-negative and is the unique solution to the fixed-point equation:

$$\beta^2 = \frac{1}{\gamma+1} G_X^2(\beta) + c(t, E_t), \;\; \beta \ge 0, \quad (9)$$

where $G_X(\beta) = \mathbb{E}\big[X_t | X_t > \beta\big] - \beta$.

Remark 8. The major differences between the problem considered in this paper and the problems considered in Gao et al. (2015a, b) are as follows: In Gao et al. (2015a, b), we had restricted the sensor to apply symmetric threshold-based policies and shown that the optimal encoding and decoding policies are piecewise affine. Furthermore, the results only hold for specific source and noise densities (e.g., Laplace source and Gamma noise with specific parameters). In this paper, however, we restrict the encoder and the decoder to apply piecewise affine encoding and decoding policies, and we show that under some weak technical assumptions (Assumptions 2 and 5), the optimal communication scheduling policy is symmetric and threshold-based. Moreover, the results hold for a large class of source densities (e.g., general even and log-concave densities).

Remark 9. Consider the case where $E_t > T - t$, that is, the sensor is always allowed to communicate with the estimator for the remaining time steps. First, we note that the opportunity cost $c(t, E_t)$ is zero. Since $G_X(0) = \mathbb{E}[X|X > 0] > 0$, the solution to (9) is not zero. Then, even though the sensor can always communicate with the estimator, the optimal communication policy is still the threshold-based policy with threshold $\beta_t^*(E_t) > 0$, which might seem counter-intuitive: why would the sensor not transmit its observation although it is allowed to do so? This surprising result is due to the fact that threshold information, i.e., whether or not the state sample belongs to a fixed, known interval, might be more informative than a noisy observation of the state at the output of the noisy channel. Hence, it might be better not to communicate explicitly over the noisy channel but rely on the side channel which signals where the sample lies. For example, at the extreme case of a very noisy channel ($\gamma \to 0$) the output of the communication channel, $\tilde{Y}_t$, is effectively useless, irrespective of the realization X_t. However, depending on the threshold and the realization X_t, thresholding information could be significantly more informative.

5. Numerical results

In this section, we present the numerical results for the problem with hard constraint. We select the source density to be Laplace density with parameter λ, i.e.,

$$p_X(x) = \begin{cases} \frac{1}{2}\lambda e^{-\lambda x}, & \text{if } x \ge 0 \\ \frac{1}{2}\lambda e^{\lambda x}, & \text{if } x < 0. \end{cases}$$

Then, it is easy to see that

$$G_X(\beta) = \mathbb{E}[X_t | X_t > \beta] - \beta = \frac{1}{\lambda}, \quad \text{for all } \beta \ge 0.$$

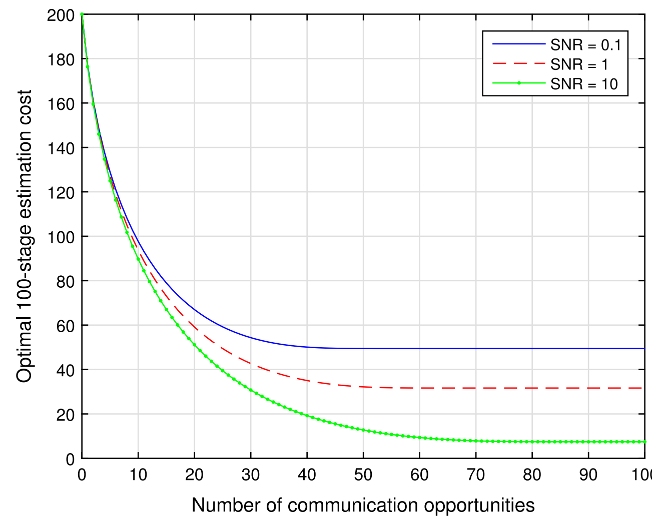

Fig. 2. 100-stage estimation error vs. the number of communication opportunities.

Hence, the solution to (9) is

$$\beta_t^*(E_t) = \sqrt{\frac{1}{\gamma+1}\frac{1}{\lambda^2} + c(t, E_t)} = \sqrt{m + c(t, E_t)},$$

where $m := 1/((\gamma+1)\lambda^2)$. Then, the optimal communication scheduling policy is described by (8). Furthermore, the optimal encoding/decoding policies (g_t^*, h_t^*) are as follows:

$$g_t(\tilde{X}_t, E_t) = \begin{cases} \alpha \cdot \big(|X_t| - \beta_t^*(E_t) - \lambda^{-1}\big), & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0 \end{cases}$$

$$h_t(\tilde{Y}_t, S_t, E_t) = \begin{cases} S_t \cdot \left(\frac{1}{\alpha}\frac{\gamma}{\gamma+1}\tilde{Y}_t + \beta_t^*(E_t) + \lambda^{-1}\right), & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0 \end{cases}$$

where $\gamma = P_T/\sigma_V^2$, and $\alpha = \sqrt{P_T/\lambda^{-2}}$. By plugging the optimal communication scheduling, encoding, and decoding policies (f_t^*, g_t^*, h_t^*) into the DP equation (6), we obtain the explicit update rule for the optimal cost-to-go $J^*(t, E_t)$, as shown below:

$$\begin{aligned} J^*(t, E_t) &= J^*(t+1, E_t) + 2\lambda^{-2}, & \text{if } E_t = 0 \\ J^*(t, E_t) &= J^*(t+1, E_t) + 2\lambda^{-2} - 2\big(\beta_t^*(E_t)\lambda^{-1} \\ &\quad + \lambda^{-2}\big)e^{-\lambda \beta_t^*(E_t)}, & \text{if } E_t > 0. \end{aligned} \quad (10)$$

We choose the parameters as follows: $T = 100$, $\lambda = 1$, and the signal-to-noise ratio (SNR) $\gamma = 0.1, 1, 10$. We solve the optimal cost-to-go $J^*(t, E_t)$ by applying the update rule (10). We have plotted the optimal 100-stage estimation error $J^*(1, N)$ versus the number of communication opportunities N under different SNRs, as shown in Fig. 2.

One can see that, for each fixed SNR, the optimal 100-stage estimation error is non-increasing in terms of the number of communication opportunities. To be more specific, there exists a threshold on the number of communication opportunities (call it *opportunity threshold*) such that the optimal 100-stage estimation error decreases when the number of communication opportunities is below the threshold, and it stays constant above the threshold. We call *minimal error* as the optimal 100-stage estimation error with the number of communication opportunities above the opportunity threshold. One can also see from Fig. 2 that when the SNR increases, the opportunity threshold increases and the minimal error decreases.

The existence of opportunity threshold was not observed in the noiseless channel setting (see Imer and Başar, 2010, Figure 5). This

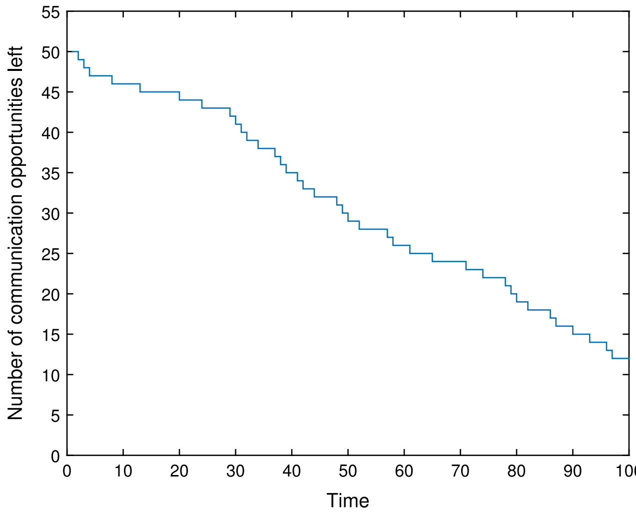

Fig. 3. A sample path of the number of remaining communication opportunities vs. time.

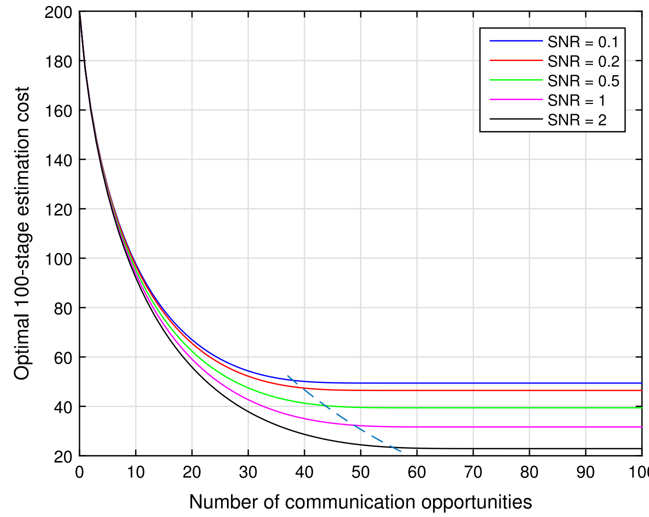

Fig. 4. Opportunity threshold vs. minimal error under different signal-to-noise ratios.

surprising phenomenon can be interpreted as follows: since the sensor applies the threshold-based policy with threshold $\beta_t^*(E_t) = \sqrt{c(t,E_t)+m} \geq \sqrt{m}$, the expectation of the consumed communication opportunities is upper bounded by $T \cdot \mathbb{P}(|X_t| \geq \sqrt{m}) = Te^{-\lambda\sqrt{m}}$. When the number of communication opportunities is greater than $Te^{-\lambda\sqrt{m}}$, the additional communication opportunities will not be consumed (in the expected sense), and thus the optimal expected estimation error will not further decrease. It can also be checked from Fig. 2 that the opportunity thresholds under different signal-to-noise ratios are roughly $Te^{-\lambda\sqrt{m}}$. Moreover, since $m = \frac{1}{\gamma+1}\frac{1}{\lambda^2}$, $Te^{-\lambda\sqrt{m}} = Te^{-1/\sqrt{\gamma+1}}$, which is an increasing function of the SNR γ. Hence, the opportunity threshold increases with the SNR.

Fig. 3 depicts a sample path of the number of remaining communication opportunities versus time. When generating the plot, we have chosen $T = 100$, $\lambda = 1$, $\gamma = 0.1$, and the number of communication opportunities $N = 50$. One can see that the communication opportunities are not used by the end of the time horizon. The reason has been discussed in Remark 9.

When the number of communication opportunities is larger than the opportunity threshold, the optimal estimation error does not change with respect to the number of communication opportunities. Without loss of generality, we can assume that the sensor is allowed to communicate at each step, that is, $N = T$. Then, the opportunity cost $c(t, E_t) = 0$. Recall that $\beta_t^*(E_t) = \sqrt{c(t,E_t)+m}$ and $m = \frac{1}{\gamma+1}\frac{1}{\lambda^2}$. Hence, the update rule for the optimal-to-go can be simplified as follows:

$$J^*(t,T) = J^*(t+1,T) + \left(\frac{2}{\lambda^2} - \left(\frac{2\sqrt{m}}{\lambda} + \frac{2}{\lambda^2}\right)\cdot e^{-\lambda\sqrt{m}}\right)$$

with $J^*(T+1,T) = 0$, which implies that

$$\begin{aligned} J^*(1,T) &= T\left(\frac{2}{\lambda^2} - \left(\frac{2\sqrt{m}}{\lambda} + \frac{2}{\lambda^2}\right)\cdot e^{-\lambda\sqrt{m}}\right) \\ &= T\,2\lambda^{-2}\left(1 - \left(\frac{1}{\sqrt{1+\gamma}} + 1\right)\cdot e^{-\frac{1}{\sqrt{1+\gamma}}}\right). \end{aligned}$$

It is straightforward to check that $J^*(1,T)$ is a decreasing function of the SNR γ. Hence, the minimal error decreases as the SNR increases.

Plotting the opportunity threshold $Te^{-\lambda m}$ versus minimal error $J^*(1,T)$ under different SNRs (dash line) in Fig. 2, we arrive at Fig. 4. One can see that the intersection between the dashed line and each solid line is roughly the turning point of the solid line. Therefore, the plot of opportunity threshold versus minimal error under different SNRs is an important one. In fact, the plot suggests the lowest capacity of the battery that one should choose when building a physical system so that the expected estimation error is minimized. In addition, the plot predicts the minimal expected estimation error.

Consider the asymptotic case where the SNR $\gamma \to \infty$, and thus $m = \frac{1}{\gamma+1}\frac{1}{\lambda^2} \to 0$. Then, the opportunity threshold $Te^{-\lambda m} \to T$, and the minimal error $J^*(1,T) \to 0$. Hence, the optimal 100-stage estimation error will be strictly decreasing in terms of the number of communication opportunities in the asymptotic case, as also noted in the prior work (see Imer and Başar, 2010, Figure 5). Moreover, the estimation error will reach zero when the number of communication opportunities is equal to the time horizon.

6. Conclusions

In this paper, we have considered a communication scheduling and remote estimation problem with a noisy communication channel. Under some technical conditions, we have obtained optimal solutions for both soft-constrained and hard-constrained problems, which consist of a symmetric threshold-based communication scheduling strategy and a pair of piecewise affine encoding/decoding strategies. Moreover, we have generated numerical results to illustrate the effect of the presence of channel noise.

There are several directions for future work. Here, we list five: (1) In this paper, we assumed that the sensor is restricted to apply the communication scheduling policy satisfying Assumption 2. Under this assumption, we showed that without loss of optimality, one can restrict the search of optimal communication scheduling policy to the symmetric class. However, it is plausible that this assumption can be relaxed or even removed, that is, for any communication scheduling policy (which may or may not satisfy Assumption 2), there exists a symmetric policy achieving no greater costs. This possible extension is not immediate and requires more effort. (2) Here, we considered the setting with a noisy channel. It will be interesting to consider a more general setting where there are two channels. One is noisy but not costly, and the other one is perfect (has high communication quality) but is costly. Then, at each time step, the sensor needs to choose whether to transmit its observation or not. If the sensor chooses to transmit its observation, it also needs to choose which channel it will use. More details on this problem can be found in Gao et al.

(2016c). (3) Here, the encoding power was taken to be time invariant. What if the encoder can distribute its total encoding power over the time horizon? More details on problems with power allocation can be found in Gao et al. (2016a, b). (4) In this paper, we considered a one-dimensional system with one sensor and one estimator. It would be interesting to consider extensions to multi-dimensional systems. To be more specific, the source input in that case would be chosen from a multi-dimensional space. In order to measure the source, we may need to place multiple sensors. Each sensor may measure the source only in one dimension, and different dimensions of the source are correlated. The sensors may send their measurements to one estimator or multiple estimators, which will produce estimate(s) on the source. Some related work on this scenario can be found in Vasconcelos and Martins (2017) and the references therein. (5) Finally, in this paper, the sensor's observations on the source were assumed to be perfect. It will be interesting to consider a more general case where there is an observation noise. Related works have been discussed in Han et al. (2015), Wu et al. (2013) and You and Xie (2013).

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